On the theory of resonances in non-relativistic QED and related models

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Abstract

We study the mathematical theory of quantum resonances in the standard model of non-relativistic QED and in Nelson's model. In particular, we estimate the survival probability of metastable states corresponding to quantum resonances and relate the resonances to poles of an analytic continuation of matrix elements of the resolvent of the quantum Hamiltonian.

1 Introduction

One of the early triumphs of Quantum Mechanics has been to enable one to calculate the discrete energy spectrum and the corresponding stationary states - eigenstates of the quantum Hamiltonian - of atoms and molecules, neglecting their interactions with the quantized electromagnetic field. However, if these interactions are taken into account, stationary states corresponding to discrete energies, save for the groundstate, are absent. The data of atomic and molecular spectroscopy can be interpreted in terms of the decay of metastable states with energies close to the discrete energies, or eigenvalues, of the non-interacting atoms or molecules. The decay of these states is accompanied by emission of photons with nearly discrete energies equal to differences between energies of stationary states; (Bohr's frequency condition). These metastable states are called "(quantum) resonances". Their analysis is the subject of this paper: We further develop some key ingredients of the mathematical theory of resonances for the standard model of "non-relativistic quantum electrodynamics" (QED) and for Nelson's model of electrons interacting with quantized (longitudinal lattice) vibrations, i.e., phonons. Due to the interactions of electrons with massless field quanta - photons or phonons - the standard techniques to analyze quantum-mechanical resonances developed during the past thirty or so years (see, e.g., [1, 2] and references therein) cannot be applied to realistic models of atoms or molecules. Our goal, in this paper, is to modify these techniques to cover the present models.

Before introducing the models we explain the resonance problem in general terms. Let H_g be a quantum Hamiltonian, where g is a real parameter called the coupling constant. Assume there is a one-parameter family of unitary transformations, $\theta \to \mathcal{U}_\theta$, with $\theta \in \mathbb{R}$, s.t. the family $H_{g,\theta} := \mathcal{U}_\theta H_g \mathcal{U}_\theta^{-1}$ has an analytic continuation in θ to a disc $D(0,\theta_0)$ in the complex plane. We call such an analytic continuation a *complex deformation* of H_g. We note that, while the essential spectrum of $H_{g,\theta}$ usually changes dramatically under such an analytic continuation, the eigenvalues are locally independent of θ, for suitably chosen deformation transformations $\mathcal{U}_\theta$, at least when they are isolated. Moreover, the real eigenvalues of $H_{g,\theta}$ coincide with the eigenvalues of H_g. The complex eigenvalues of $H_{g,\theta}$, $\mathrm{Im}\theta > 0$, are called the (quantum $\mathcal{U}-$)resonance eigenvalues - or just resonance eigenvalues - of the Hamiltonian H_g. The transformations most commonly used is the

group of dilatations of positions and momenta (see below), and the corresponding resonances are sometimes called “dilatation resonances”.

It is plausible from our definition that resonances - at least for weakly coupled systems (g small) - are closely related to eigenvalues of $H_{g=0}$. But what is their physical significance?

Physically, one thinks of quantum resonances as long-lived metastable states or as “bumps” in the scattering cross-section as a function of energy. The energies and life-times of metastable states are given by the bumps’ centers and the inverse of the bumps’ widths. A known approach to establish such properties is as follows. Let $\mathcal{D} \subset \mathcal{H}$ denote the dense linear subspace of $\mathcal{U}-$entire vectors, i.e., vectors ψ for which the family $\{\psi_\theta := \mathcal{U}_\theta \psi\}_{\theta \in \mathbb{R}}$ has an analytic continuation to the entire complex plane. For such vectors one has the “Combes formula”

$$(\psi, (H_g - z)^{-1}\psi) = (\psi_{\bar{\theta}}, (H_{g,\theta} - z)^{-1}\psi_\theta). \tag{1.1}$$

If we continue the r.h.s. analytically, first in θ and then in z, then we see that matrix elements, $(\psi, (H_g - z)^{-1}\psi)$, of the resolvent, for $z \in \mathbb{C}$, $\mathrm{Im} z > 0$, and $\psi \in \mathcal{D}$, have an analytic continuation in z across the essential spectrum of H_g to the “second Riemann sheet” whenever the resolvent set of the operator $H_{g,\theta}$, $\mathrm{Im}\theta > 0$, contains a part of this essential spectrum[1]. Clearly, eigenvalues of $H_{g,\theta}$, $\mathrm{Im}\theta > 0$, in the lower complex half-plane, $\mathbb{C}^-$, are poles of this analytic continuation, provided these eigenvalues are isolated.

In the latter case, the metastability property can be established (at least, for weakly coupled systems) by using the relation - via the Fourier transform - between the propagator and the resolvent, contour deformation and Cauchy’s theorem (see [3, 1]). The “bumpiness” of the cross-section can be connected to the resonance poles. The real and imaginary parts of the resonance eigenvalues give the energy and the rate of decay, or the reciprocal life-time, of the metastable state.

The situation described above is exactly the one encountered in Quantum Mechanics. In non-relativistic QED and phonon models, the resonance eigenvalues are *not isolated*; more precisely, a branch of essential spectrum is attached to every complex eigenvalue of the deformed Hamiltonian $H_{g,\theta}$. This is due to the fact that photons and phonons are *massless*. As a result, establishing the property of metastability and the pole structure of the resolvent (and the related bumpiness of the cross-section) becomes a challenge. In this paper, we prove, for non-relativistic QED and Nelson’s model, the metastability property of resonances and characterize them in terms of poles of a meromorphic continuation of the matrix elements of the resolvent on a dense set of vectors.

Next, we introduce the models considered in this paper. The Hamiltonian of the QED model is defined as

$$H_g^{SM} := \sum_{j=1}^{N} \frac{1}{2m_j}(p_j + gA(x_j))^2 + V(x) + H_f, \tag{1.2}$$

where $x = (x_1, \dots, x_N)$, $p_j = -i\nabla_j$ denotes the momentum of the j^{th} particle and m_j its mass, and $V(x)$ is the potential energy of the particle system. Furthermore, $A(y)$ denotes the quantized vector potential

$$A(y) = \sum_{\lambda \in \{-1,1\}} \int \frac{d^3k}{(2\pi)^3} \frac{\chi(k)}{\sqrt{2|k|}} (e^{ik\cdot y}\epsilon_\lambda(k)a_\lambda(k) + e^{-ik\cdot y}\overline{\epsilon_\lambda(k)}a_\lambda^*(k)), \tag{1.3}$$

[1] Here we use the terms Riemann sheet and Riemann surface informally. However, we expect that the matrix elements (1.1) do have a Riemann surface ramified at the resonances of H_g.

where $k \in \mathbb{R}^3$, χ is an ultraviolet cut-off that vanishes sufficiently fast at infinity, and $\epsilon_\lambda(k), \lambda = -1, 1$, are two transverse polarization vectors, i.e., orthonormal vectors in $\mathbb{R}^3 \otimes \mathbb{C}$ satisfying $k \cdot \epsilon_\lambda(k) = 0$; moreover, H_f is the photon (quantized electromagnetic field) Hamiltonian defined as

$$H_f = \sum_{\lambda=-1,1} \int_{\mathbb{R}^3} \omega(k) a_\lambda^*(k) a_\lambda(k) dk, \tag{1.4}$$

where $\omega(k) = |k|$.

The operator-valued distributions $a_\lambda(k)$ and $a_\lambda^*(k)$ are annihilation and creation operators acting on the symmetric Fock space $\mathcal{F}_s$ over $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. They obey the canonical commutation relations

$$[a_\lambda^*(k), a_{\lambda'}^*(k')] = [a_\lambda(k), a_{\lambda'}(k')] = 0 \quad , \quad [a_\lambda(k), a_{\lambda'}^*(k')] = \delta(k-k')\delta_{\lambda,\lambda'}, \tag{1.5}$$

and

$$a_\lambda(k)\Omega = 0,$$

where $\Omega \in \mathcal{F}_s$ is the vacuum vector.

The QED Hamiltonian H_g^{SM} acts on the Hilbert space $\mathcal{H}_p \otimes \mathcal{F}_s$, where $\mathcal{H}_p$ is the Hilbert space for N electrons, e.g. $\mathcal{H}_p = \mathrm{L}^2(\mathbb{R}^{3N})$, (neglecting permutation symmetry). In (1.2), Zeeman terms coupling the magnetic moments of the electrons to the magnetic field are neglected.

Nelson's model describes non-relativistic particles without spin interacting with a scalar, massless boson field. The Hamiltonian of the model acts on $\mathcal{H}_p \otimes \mathcal{F}_s$, where $\mathcal{F}_s$ is the symmetric Fock space over $\mathrm{L}^2(\mathbb{R}^3)$, and is given by

$$H_g^N := H_p \otimes I + I \otimes H_f + W_g. \tag{1.6}$$

Here, $H_p = \sum_{j=1}^N p_j^2/2m_j + V$ denotes an N-particle Schrödinger operator on $\mathcal{H}_p$. We assume that its spectrum, $\sigma(H_p)$, consists of a sequence of discrete eigenvalues, $\lambda_0, \lambda_1, \cdots$, below some real number Σ called the *ionization threshold*.

For k in $\mathbb{R}^3$, we denote by $a^*(k)$ and $a(k)$ the usual phonon creation and annihilation operators on $\mathcal{F}_s$. They are operator-valued distributions obeying the canonical commutation relations

$$[a^*(k), a^*(k')] = [a(k), a(k')] = 0 \quad , \quad [a(k), a^*(k')] = \delta(k-k'). \tag{1.7}$$

The operator associated with the energy of the free boson field, H_f, is given by the expression (1.4), except that the operators $a^*(k)$ and $a(k)$ now are scalar creation and annihilation operators as given above. The interaction W_g in (1.6) is assumed to be of the form

$$W_g = g\phi(G_x) \tag{1.8}$$

where

$$\phi(G_x) = \sum_{j=1}^N \int_{\mathbb{R}^3} \frac{\chi(k)}{|k|^{1/2-\mu}} \left[e^{-ik\cdot x_j} a^*(k) + e^{ik\cdot x_j} a(k) \right] dk. \tag{1.9}$$

As above, the function $\chi(k)$ denotes an ultraviolet cut-off, and the parameter μ is assumed to be positive.

Next, we state our assumptions on the potential and the ultraviolet cut-off χ, in particular concerning analyticity under dilatations.

(A) The potential $V(x)$ is dilatation analytic, i.e., the vector-function $\theta \mapsto V(e^\theta x)(-\Delta+1)^{-1}$ has an analytic continuation to a small complex disc $D(0,\theta_0) \subset \mathbb{C}$, for some $\theta_0 > 0$.

An example of a dilatation-analytic potential V is the Coulomb potential for N electrons and one fixed nucleus located at the origin. For a molecule in the Born-Oppenheimer approximation, the potential $V(x)$ is not dilatation-analytic. In this case, one has to use a more general notion of distortion analyticity (see [1]), which can be easily accommodated in our analysis.

(B) The function χ is dilatation analytic, i.e., $\theta \mapsto \chi(e^{-\theta}k)$ has an analytic continuation from $\mathbb{R}$ to the disc $D(0,\theta_0)$.

For instance, we can choose $\chi(k) := e^{-k^2/\Lambda^2}$, for some fixed, arbitrarily large ultraviolet cut-off $\Lambda > 0$.

Let H_g denote either H_g^{SM} or H_g^N. To define quantum resonances for the Hamiltonian H_g, we use the dilatations of electron positions and of photon momenta:

$$x_j \to e^\theta x_j \text{ and } k \to e^{-\theta}k,$$

where θ is a real parameter. Such dilatations are represented by the one-parameter group of unitary operators, $\mathcal{U}_\theta$, on the total Hilbert space $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}_s$ of the system. This is one of the most important examples of the deformation groups mentioned above[2]. Following the general prescription, we define, for $\theta \in \mathbb{R}$, the family of unitarily equivalent Hamiltonians

$$H_{g,\theta} := \mathcal{U}_\theta H_g \mathcal{U}_\theta^{-1}. \tag{1.10}$$

By the above assumptions on V and χ, the family $H_{g,\theta}$ can be analytically extended, as a type-A family in the sense of Kato, to all θ belonging to the disc $D(0,\theta_0)$ in the complex plane, where θ_0 is as in assumptions (A) and (B). The deformation resonances are now defined as complex eigenvalues of $H_{g,\theta}$, $\mathrm{Im}\theta > 0$.

Let $\lambda_0 := \inf(\sigma(H_{g=0}))$. We consider the eigenvalues λ_j of H_p, or of $H_0 := H_p \otimes I + I \otimes H_f$, with $\lambda_0 < \lambda_j < \Sigma$. By the renormalization group analysis in [4, 5, 6, 7], we know that, as the interaction between the non-relativistic particles and the field is turned on, these eigenvalues turn into resonances $\lambda_{j,g}$, with $\mathrm{Im}\lambda_{j,g} < 0$ and these resonances are $\theta-$independent; (see also [8] for a somewhat different model). Our goal is to investigate the properties of these resonances, as described above.

To simplify our presentation, we assume that λ_j is non-degenerate, and we denote by $\Psi_j = \psi_j \otimes \Omega$ the normalized, unperturbed eigenstate associated with λ_j. We also assume that

(C) *Fermi's Golden Rule* ([4, 5, 6]) holds.

This condition implies that $\mathrm{Im}\lambda_{j,g} \leq -\mathrm{c}_0 g^2$, for some positive constant c_0; see for example [4, 5, 6].

The main results of this paper are summarized in the following theorems.

Theorem 1.1 *Let H_g be either H_g^{SM} or H_g^N. Given Ψ_j, and $\lambda_{j,g}$ as above, and under Assumptions (A)-(C) formulated above, there exists some $g_0 > 0$ such that, for all $0 < g < g_0$ and times $t \geq 0$,*

$$\left(\Psi_j, e^{-itH_g}\Psi_j\right) = e^{-it\lambda_{j,g}} + O(g^\alpha), \tag{1.11}$$

where $\alpha := \frac{2+4\mu}{5+2\mu}$, with $\mu > 0$ appearing in (1.9) for the Nelson model, and $\alpha = \frac{2}{3}$ for QED.

[2]See, however, Remark 6.2 on page 25.

Remark 1.2 *We expect that our approach extends to situations where Fermi's Golden Rule condition fails, as long as* $\mathrm{Im}\lambda_{j,g} < 0$*, and that we can improve the exponent of* g *in the error term by using an initial state that is a better approximation of the "resonance state"; see section 3.*

Remark 1.3 *The analysis below, together with Theorem 3.3 in [9], gives an adiabatic theorem for quantum resonances in non-relativistic QED.*

Theorem 1.1 estimates the survival probability, $\left(\Psi_j, e^{-itH_g}\Psi_j\right)$, of the state Ψ_j. Let $\gamma_{j,g} := -\mathrm{Im}\lambda_{j,g}$ and $T_{j,g} := 1/\gamma_{j,g}$. Theorem 1.1 implies that

$$\|e^{-itH_g}\Psi_j - e^{-it\lambda_{j,g}}\Psi_j\| = [1 - e^{-2t\gamma_{j,g}} + O(g^\alpha)]^{1/2}, \tag{1.12}$$

which is $\ll 1$, for $t \ll T_{j,g}$. This property is what we call the *"metastability"* of the resonance associated with the resonance eigenvalue $\lambda_{j,g}$.

There is a dense linear subspace $\mathcal{D} \subset \mathcal{H}$ of vectors with the property that, for every $\psi \in \mathcal{D}$, the family $\{\mathcal{U}_\theta\psi\}_{\theta\in\mathbb{R}}$ of vectors has an analytic extension in θ to the entire complex plane, with $\mathcal{U}_\theta\psi \in \mathcal{D}$, for any $\theta \in \mathbb{C}$. Vectors in $\mathcal{D}$ are called dilatation-entire vectors.

Next, for $z_* \in \mathbb{C}$ and $0 \le \varphi_1 < \varphi_2 < 2\pi$, we define domains

$$W_{z_*}^{\varphi_1,\varphi_2} := \{z \in \mathbb{C} | \ \ |z - z_*| < \frac{1}{2}|\mathrm{Im} z_*|, \ \varphi_1 \le \arg(z - z_*) \le \varphi_2\}.$$

Our second main result is the following theorem.

Theorem 1.4 *Let* H_g *be either* H_g^{SM} *or* H_g^N*. Let Conditions (A), (B) and (C) be satisfied, and let* $\lambda_0 < \lambda_j < \Sigma$ *be an eigenvalue of* H_p*. Then there are a constant* $g_* > 0$ *and a dense set* $\mathcal{D}' \subset \mathcal{D}$ *s.t., for* $g < g_*$ *and for all* $\psi \in \mathcal{D}'$*, the function*

$$F_\psi(z) := (\psi, (H_g - z)^{-1}\psi)$$

has an analytic continuation in z *from the upper half-plane, across a neighbourhood of* λ_j*, into the domain* $W_{\lambda_{j,g}}^{\varphi_1,\varphi_2}$*, for some* $\varphi_1 < \pi/2$ *and* $\varphi_2 > \pi$*, and this continuation satisfies the relations*

$$F_\psi(z) = \frac{p(\psi)}{\lambda_{j,g} - z} + r(z;\psi), \tag{i}$$

with

$$|r(z;\psi)| \le \mathrm{C}(\psi)|\lambda_{j,g} - z|^{-\beta}, \tag{ii}$$

for some $\beta < 1$*. Here* $p(\psi)$ *and* $r(z;\psi)$ *are quadratic forms on the domain* $\mathcal{D}' \times \mathcal{D}'$*.*

Remark 1.5 *Since we can rotate the essential spectrum of* $H_{g,\theta}, \theta \in D(0,\theta_0)$*, in* $\mathbb{C}^-$ *using dilatation analyticity, if* $\theta_0 < \frac{\pi}{2}$ *is large enough, we expect that* $F_\psi(z)$ *can be analytically continued in* z *from the upper half-plane into a neighbourhood of* $\lambda_{j,g}$ *that is larger than* $W_{\lambda_{j,g}}^{\varphi_1,\varphi_2}$ *given in Theorem 1.4. In this case the quadratic form* $r(w;\psi)$ *would also depend on the homotopy class of the path along which* $F_\psi(z)$ *is analytically continued from the upper half-plane to the point* w *in the vicinity of* $\lambda_{j,g}$*.*

For an operator A on the one-particle space $L^2(\mathbb{R}^3)$, we denote by $d\Gamma(A)$ its "lifting" to the Fock space $\mathcal{F}_s$, (second quantization). The set $\mathcal{D}'$ in Theorem 1.4 can be chosen explicitly as

$$\mathcal{D}' := \{\psi \in \mathcal{D} | \ \|d\Gamma(\omega^{-1/2})(1 - P_\Omega)\psi\| < \infty\},$$

where P_Ω is the projection onto the vacuum Ω in $\mathcal{F}_s$, for the Nelson model. In this case $\beta = (1 + \frac{2}{3}\mu)^{-1}$. For QED, we define

$$\mathcal{D}' := \{\psi \in \mathcal{D} | \ \|e^{\delta\langle x\rangle} d\Gamma(\omega^{-1/2})(1 - P_\Omega)\psi\| < \infty \text{ for some } \delta > 0\}.$$

Since $\mathcal{U}_\theta d\Gamma(\omega^{-1/2}) = e^{\theta/2} d\Gamma(\omega^{-1/2})\mathcal{U}_\theta$, the set $\mathcal{D}'$ is dense in $\mathcal{D}$.

The main difficulty in the proofs of our main results comes from the fact that the unperturbed eigenvalue λ_j is the threshold of a branch of continuous spectrum. To overcome this difficulty, we introduce an infrared cut-off that opens a gap in the spectrum of $H_{g,\theta}$, and we control the error introduced by opening such a gap using the fact that the interaction between the electrons and the photons or phonons vanishes sufficiently fast at low photon/phonon energies (see [4, 5, 10] and Eqn. (2.7) below).

Our paper is organized as follows. In Sections 2-4 we prove Theorem 1.1 for the Nelson Hamiltonian, H_g^N. In Section 5 we extend this proof to the QED Hamiltonian, H_g^{SM}. Theorem 1.4 is proven in Section 6.

As we were completing this paper, there appeared an e-print [11] where lower and upper bounds for the lifetime of the metastable states considered in this paper are established by somewhat different techniques.

Acknowledgements. J.Fr. and I.M.S. would like to thank M. Griesemer for many useful discussions on related problems. J.Fa. is grateful to I.M.S. and W.A.S. for hospitality at the University of Toronto and I.M.S., and I.M.S. and W.A.S. are grateful to J.Fr. for hospitality at ETHZ.

2 Dilatation analyticity and IR cut-off Hamiltonians

Let $H_g = H_g^N$ be the Hamiltonian defined in (1.6). We begin this section with a discussion of the dilatation deformation $H_{g,\theta}$ of H_g defined in the introduction, Eqn (1.10). As was already mentioned above, by the above assumptions on V and χ, the family $H_{g,\theta}$ can be analytically extended to all θ belonging to a disc $D(0, \theta_0)$ in the complex plane. The relation $H_{g,\theta}^* = H_{g,\overline{\theta}}$ holds for real θ and extends by analyticity to $\theta \in D(0, \theta_0)$. A direct computation gives

$$H_{g,\theta} = H_{p,\theta} \otimes I + e^{-\theta} I \otimes H_f + W_{g,\theta},$$

where $H_{p,\theta} = \mathcal{U}_\theta H_p \mathcal{U}_\theta^{-1}$ and $W_{g,\theta} := \mathcal{U}_\theta W_g \mathcal{U}_\theta^{-1}$. Note that $W_{g,\theta} = g\phi(G_{x,\theta})$, with

$$G_{x,\theta}(k) = e^{-(1+\mu)\theta} \frac{\chi(e^{-\theta}k)}{|k|^{1/2-\mu}} e^{-ik.x}. \tag{2.1}$$

We now introduce an infra-red cut-off Hamiltonian

$$H_{g,\theta}^\sigma := H_{p,\theta} \otimes I + e^{-\theta} I \otimes H_f + W_{g,\theta}^{\geqslant\sigma}, \tag{2.2}$$

where $W^{\geqslant\sigma}_{g,\theta} := g\phi(G^{\leqslant\sigma}_{x,\theta})$, and $G^{\leqslant\sigma}_{x,\theta} := \kappa_\sigma G_{x,\theta}$. Here κ_σ is an infrared cut-off function that we can choose, for instance, as $\kappa_\sigma = \mathbf{1}_{|k|\geq\sigma}$. We also define

$$W^{\leqslant\sigma}_{g,\theta} := W_{g,\theta} - W^{\geqslant\sigma}_{g,\theta} = g\phi(G^{\geqslant\sigma}_{x,\theta}), \tag{2.3}$$

where $G^{\geqslant\sigma}_{x,\theta} := (1-\kappa_\sigma)G_{x,\theta}$. We then have that

$$H_{g,\theta} = H^{\sigma}_{g,\theta} + W^{\leqslant\sigma}_{g,\theta}. \tag{2.4}$$

We denote by $\mathcal{F}_s^{\geqslant\sigma}$ and $\mathcal{F}_s^{\leqslant\sigma}$ the symmetric Fock spaces over $\mathrm{L}^2(\{k \in \mathbb{R}^3 : |k| \geq \sigma\})$ and $\mathrm{L}^2(\{k \in \mathbb{R}^3 : |k| \leq \sigma\})$, respectively. It is well-known that there exists a unitary operator $\mathcal{V}$ that maps $\mathrm{L}^2(\mathbb{R}^{3N}; \mathcal{F}_s)$ to $\mathrm{L}^2(\mathbb{R}^{3N}; \mathcal{F}_s^{\geqslant\sigma}) \otimes \mathcal{F}_s^{\leqslant\sigma}$, so that

$$\mathcal{V} H^{\sigma}_{g,\theta} \mathcal{V}^{-1} = H^{\geqslant\sigma}_{g,\theta} \otimes I + e^{-\theta} I \otimes H_f^{\leqslant\sigma}. \tag{2.5}$$

Here, $H^{\geqslant\sigma}_{g,\theta}$ acts on $\mathrm{L}^2(\mathbb{R}^{3N}; \mathcal{F}_s^{\geqslant\sigma})$ and is defined by

$$H^{\geqslant\sigma}_{g,\theta} := H_{p,\theta} + e^{-\theta} H_f^{\geqslant\sigma} + W^{\geqslant\sigma}_{g,\theta}. \tag{2.6}$$

The operators $H_f^{\geqslant\sigma}$ and $H_f^{\leqslant\sigma}$ denote the restrictions of H_f to $\mathcal{F}_s^{\geqslant\sigma}$ and $\mathcal{F}_s^{\leqslant\sigma}$ respectively. The unitary operator $\mathcal{V}$ will be sometimes dropped in the sequel if no confusion may arise. We note the following estimate that will often be used in this paper:

$$\left\| W^{\leqslant\sigma}_{g,\theta} \left[H_f + 1\right]^{-1} \right\| \leq \mathrm{C} g \sigma^{1/2+\mu}, \tag{2.7}$$

where $\mu > 0$, $\;$ C is a positive constant, and $\theta \in D(0, \theta_0)$.

We now consider an unperturbed isolated eigenvalue λ_j of H_0. To simplify our analysis, we assume that λ_j is non-degenerate. Let

$$\mathrm{d}_j := \mathrm{dist}(\lambda_j; \sigma(H_p)\backslash\{\lambda_j\}), \tag{2.8}$$

which is positive. It is shown in [4, 5, 12] that, as the perturbation W_g is turned on, the eigenvalue λ_j turns into a resonance $\lambda_{j,g}$ of H_g. In other words, for $\theta \in D(0, \theta_0)$ with $\mathrm{Im}(\theta) > 0$, there exists a non-degenerate eigenvalue $\lambda_{j,g}$ of $H_{g,\theta}$ *not* depending on θ, with $\mathrm{Re}\lambda_{j,g} = \lambda_j + O(g^2)$, $\mathrm{Im}\lambda_{j,g} = O(g^2)$, and, if Fermi's Golden Rule condition holds, $\mathrm{Im}\lambda_{j,g} \leq -\mathrm{c}_0 g^2$, for some positive constant c_0. Similarly, the operator $H^{\geqslant\sigma}_{g,\theta}$ has an eigenvalue $\lambda^{\geqslant\sigma}_{j,g}$ bifurcating from the eigenvalue λ_j of H_0 having the same properties as $\lambda_{j,g}$, with the important exception that $\lambda^{\geqslant\sigma}_{j,g}$ depends on θ. The reason for this is that $H^{\geqslant\sigma}_{g,\theta+r} \neq \mathcal{U}_r H^{\geqslant\sigma}_{g,\theta} \mathcal{U}_{-r}$, $\;r \in \mathbb{R}$. Furthermore, we have the crucial property (see Proposition 4.1) that the eigenvalue $\lambda^{\geqslant\sigma}_{j,g}$ of $H^{\geqslant\sigma}_{g,\theta}$ is isolated from the rest of the spectrum of $H^{\geqslant\sigma}_{g,\theta}$. More precisely,

$$\mathrm{dist}\left(\lambda^{\geqslant\sigma}_{j,g}, \sigma(H^{\geqslant\sigma}_{g,\theta}) \setminus \{\lambda^{\geqslant\sigma}_{j,g}\}\right) \geq \mathrm{C}\sigma, \tag{2.9}$$

for some positive constant C independent of σ.

It is tempting to treat $H_{g,\theta}$ as a perturbation of $H_{g,\theta}^{\geqslant\sigma}$. However, we have to take care of the difference between $\lambda_{j,g}$ and $\lambda_{j,g}^{\geqslant\sigma}$. In order to deal with this problem, we "renormalize" the unperturbed part $H_{g,\theta}^{\sigma}$ by setting

$$\widetilde{H}_{g,\theta}^{\sigma} = H_{g,\theta}^{\sigma} + \left(\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}\right)\mathcal{V}^{-1}(P_{g,\theta}^{\geqslant\sigma} \otimes I)\mathcal{V}. \tag{2.10}$$

Here $P_{g,\theta}^{\geqslant\sigma}$ denotes the spectral projection onto the eigenspace associated with the eigenvalue $\lambda_{j,g}^{\geqslant\sigma}$ of $H_{g,\theta}^{\geqslant\sigma}$. As in (2.5), we have the representation

$$\mathcal{V}\widetilde{H}_{g,\theta}^{\sigma}\mathcal{V}^{-1} = \widetilde{H}_{g,\theta}^{\geqslant\sigma} \otimes I + e^{-\theta} I \otimes H_f^{\leqslant\sigma}, \tag{2.11}$$

where we have set

$$\widetilde{H}_{g,\theta}^{\geqslant\sigma} = H_{g,\theta}^{\geqslant\sigma} + \left(\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}\right) P_{g,\theta}^{\geqslant\sigma}. \tag{2.12}$$

By (2.12), we see that $\lambda_{j,g}$ is a non-degenerate eigenvalue of $\widetilde{H}_{g,\theta}^{\geqslant\sigma}$. In Proposition 6.3 we will show that there exists a positive constant C such that

$$\left|\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}\right| \leq \mathrm{C} g^2 \sigma^{1+\mu}, \tag{2.13}$$

and that the operator $\widetilde{H}_{g,\theta}^{\geqslant\sigma}$ still has a gap of order $O(\sigma)$ around $\lambda_{j,g}$. Then the decomposition (2.2) is replaced by

$$H_{g,\theta} = \widetilde{H}_{g,\theta}^{\sigma} + \widetilde{W}_{g,\theta}^{\leqslant\sigma}, \tag{2.14}$$

where

$$\widetilde{W}_{g,\theta}^{\leqslant\sigma} = W_{g,\theta}^{\leqslant\sigma} - \left(\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}\right)\mathcal{V}^{-1} P_{g,\theta}^{\geqslant\sigma} \otimes I\mathcal{V}. \tag{2.15}$$

Let $H_\star^{\#}$ denote one of the operators H_g, $H_{g,\theta}$, $H_{g,\theta}^{\sigma}$ or $H_{g,\theta}^{\geqslant\sigma}$. We write its resolvent by using the notation $R_\star^{\#}(z) = \left[H_\star^{\#} - z\right]^{-1}$. Similarly, we define $\widetilde{R}_\star^{\#}(z) = \left[\widetilde{H}_\star^{\#} - z\right]^{-1}$.

3 Proof of Theorem 1.1

We begin with some notation. We consider an interval I of size δ, containing λ_j, such that $\delta < \frac{1}{2}\mathrm{d}_j$. For concreteness, let

$$\mathrm{I} = \left(\lambda_j - \frac{\delta}{2}, \lambda_j + \frac{\delta}{2}\right). \tag{3.1}$$

Define, in addition,

$$\mathrm{I}_1 = \left(\lambda_j - \frac{\delta}{4}, \lambda_j + \frac{\delta}{4}\right). \tag{3.2}$$

We consider a smooth function $f \in \mathrm{C}_0^\infty(\mathrm{I})$, $\mathrm{Ran}(f) \in [0,1]$, such that $f = 1$ on I_1. It is known that there exists an almost analytic extension $\widetilde{f}$ of f such that

$$\widetilde{f} = 1 \text{ on } \{z \in \mathbb{C}|\ \ \mathrm{Re}(z) \in \mathrm{I}_1\} \quad , \quad \mathrm{supp}(\widetilde{f}) \subset \{z \in \mathbb{C}|\ \ \mathrm{Re}(z) \in \mathrm{I}\}, \tag{3.3}$$

and $\left|(\partial_{\bar{z}}\widetilde{f})(z)\right| = O(\delta^{-1}|\mathrm{Im}(z)/\delta|^n)$, for any $n \in \mathbb{N}$. We shall use these properties of $\widetilde{f}$ in the sequel.

We begin with the following proposition.

Proposition 3.1 *Given H_g, Ψ_j, $\lambda_{j,g}$ and f as above, there exists $g_0 > 0$ such that, for all $0 < g \leq g_0$, $\delta = \mathrm{C}\sigma$, $\mathrm{C} > 1$, and $\sigma = g^{2-\frac{2+4\mu}{5+2\mu}}$,*

$$\left(\Psi_j, e^{-itH_g} f(H_g)\Psi_j\right) = e^{-it\lambda_{j,g}} + O(g^{\frac{2+4\mu}{5+2\mu}}), \tag{3.4}$$

for all times $t \geq 0$.

We divide the proof of Proposition 3.1 into several steps, deferring the proof of some technical ingredients to the following section. We extend a method due to Hunziker to prove Proposition 3.1, see [3] or [1]. Let $\mathcal{N}(\theta)$ be a punctured neighbourhood of λ_j such that $\mathcal{N}(\theta) \cap \sigma(\widetilde{H}^{\geqslant\sigma}_{g,\theta}) = \lambda_{j,g}$ and $\mathrm{I} \subset \mathcal{N}(\theta) \cup \{\lambda_j\}$. Let $\Gamma \subset \mathcal{N}(\theta)$ be a contour that encloses I and $\lambda_{j,g}$. For z inside Γ, we have that

$$\widetilde{R}^{\geqslant\sigma}_{g,\theta}(z) = \frac{P^{\geqslant\sigma}_{g,\theta}}{\lambda_{j,g} - z} + \widehat{R}^{\geqslant\sigma}_{g,\theta}(z), \tag{3.5}$$

where $P^{\geqslant\sigma}_{g,\theta}$ denotes the spectral projection onto the eigenspace associated to the eigenvalue $\lambda_{j,g}$ of $\widetilde{H}^{\geqslant\sigma}_{g,\theta}$, that is

$$P^{\geqslant\sigma}_{g,\theta} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \widetilde{R}^{\geqslant\sigma}_{g,\theta}(z)dz, \tag{3.6}$$

where $\mathcal{C}$ denotes a circle centered at $\lambda_{j,g}$ with radius chosen so that $\mathcal{C} \subset \rho(H^{\geqslant\sigma}_{g,\theta}) \cap \mathcal{N}(\theta)$, and the *regular part*, $\widehat{R}^{\geqslant\sigma}_{g,\theta}(z)$, is given by

$$\widehat{R}^{\geqslant\sigma}_{g,\theta}(z) := \widetilde{R}^{\geqslant\sigma}_{g,\theta}(z)(1 - P^{\geqslant\sigma}_{g,\theta}) = R^{\geqslant\sigma}_{g,\theta}(z)(1 - P^{\geqslant\sigma}_{g,\theta}), \tag{3.7}$$

which can be written as

$$\widehat{R}^{\geqslant\sigma}_{g,\theta}(z) = \frac{1}{2\pi i}\oint_{\Gamma} \widetilde{R}^{\geqslant\sigma}_{g,\theta}(w)(w-z)^{-1}dw, \tag{3.8}$$

where z is inside Γ. Note that

$$\widehat{R}^{\geqslant\sigma}_{g,\theta} P^{\geqslant\sigma}_{g,\theta} = P^{\geqslant\sigma}_{g,\theta} \widehat{R}^{\geqslant\sigma}_{g,\theta} = 0, \tag{3.9}$$

and

$$(P^{\geqslant\sigma}_{g,\theta})^2 = P^{\geqslant\sigma}_{g,\theta}. \tag{3.10}$$

We will need the following easy lemma, which follows from dilatation analyticity and Stone's theorem.

Lemma 3.2 *Assume that the infrared cut-off parameter σ is chosen such that $g^2 \ll \sigma < g^{\frac{3}{2+\mu}} \ll 1$. Then*

$$\left(\Psi_j, e^{-itH_g} f(H_g)\Psi_j\right) = A(t,\overline{\theta}) - A(t,\theta) + B(t,\overline{\theta}) - B(t,\theta), \tag{3.11}$$

for $\theta \in D(0,\theta_0)$, $\mathrm{Im}\theta > 0$, where

$$A(t,\theta) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-itz} f(z)\left(\Psi_{j,\overline{\theta}}, \widetilde{R}^{\sigma}_{g,\theta}(z)\Psi_{j,\theta}\right)dz, \tag{3.12}$$

$$B(t,\theta) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-itz} f(z)\left(\Psi_{j,\overline{\theta}}, \widetilde{R}^{\sigma}_{g,\theta}(z)\sum_{n\geq 1}\left(-\widetilde{W}^{\leqslant\sigma}_{g,\theta}\widetilde{R}^{\sigma}_{g,\theta}(z)\right)^n \Psi_{j,\theta}\right)dz. \tag{3.13}$$

Proof. By Stone's theorem,

$$\left(\Psi_j, e^{-itH_g} f(H_g)\Psi_j\right) = \lim_{\varepsilon\searrow 0}\frac{1}{2\pi i}\int_{\mathbb{R}} e^{-itz} f(z)\left(\Psi_j, \left[R_g(z-i\varepsilon) - R_g(z+i\varepsilon)\right]\Psi_j\right) dz. \tag{3.14}$$

Since H_g and Ψ_j are dilatation analytic, this implies for $\theta \in D(0,\theta_0)$

$$\left(\Psi_j, e^{-itH_g} f(H_g)\Psi_j\right) = F(t,\overline{\theta}) - F(t,\theta), \tag{3.15}$$

where

$$F(t,\theta) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-itz} f(z)\left(\Psi_{j,\overline{\theta}}, R_{g,\theta}(z)\Psi_{j,\theta}\right) dz. \tag{3.16}$$

It follows from Lemma 4.4, below, that we can expand $R_{g,\theta}(z)$ into a Neumann series, which is convergent under our assumptions on g and σ if Fermi's Golden Rule holds. We obtain

$$F(t,\theta) = A(t,\theta) + B(t,\theta), \tag{3.17}$$

for $\theta \in D(0,\theta_0)$, $\mathrm{Im}\theta > 0$, and hence the claim of the lemma is proven. $\square$

In what follows, we fix $\theta \in D(0,\theta_0)$ with $\mathrm{Im}\theta > 0$. We estimate $A(t,\overline{\theta}) - A(t,\theta)$ and $B(t,\overline{\theta}) - B(t,\theta)$ in the following two lemmata.

Lemma 3.3 *For $g^2 \ll \sigma < \delta \ll 1$, we have*

$$A(t,\overline{\theta}) - A(t,\theta) = e^{-it\lambda_{j,g}} + O(\delta g^2\sigma^{-2}) + O(g^2\sigma^{-1}),$$

for all $t \geq 0$.

Proof. It follows from the spectral theorem that

$$\mathcal{V}\widetilde{R}^{\sigma}_{g,\theta}(z)\mathcal{V}^{-1} = \int_{\sigma(H_f^{\leqslant\sigma})} \widetilde{R}^{\geqslant\sigma}_{g,\theta}(z - e^{-\theta}\omega) \otimes dE_{H_f^{\leqslant\sigma}}(\omega), \tag{3.18}$$

where $E_{H_f^{\leqslant\sigma}}$ are the spectral projections of $H_f^{\leqslant\sigma}$; see for example [2]. Furthermore, $\mathcal{V}\Psi_{j,\theta} = \psi_{j,\theta}\otimes\Omega^{\geqslant\sigma}\otimes\Omega^{\leqslant\sigma}$, where $\Omega^{\geqslant\sigma}$ (respectively $\Omega^{\leqslant\sigma}$) denotes the vacuum in $\mathcal{F}_s^{\geqslant\sigma}$ (in $\mathcal{F}_s^{\leqslant\sigma}$). Inserting this into (3.12) and using (3.18), we get

$$A(t,\theta) = \frac{1}{2i\pi}\int_{\mathbb{R}} e^{-itz} f(z)\left(\psi_{j,\overline{\theta}}\otimes\Omega^{\geqslant\sigma}, \widetilde{R}^{\geqslant\sigma}_{g,\theta}(z)\psi_{j,\theta}\otimes\Omega^{\geqslant\sigma}\right) dz. \tag{3.19}$$

¿From Proposition 4.1, we know that the spectrum of $H^{\geqslant\sigma}_{g,\theta}$ is of the form pictured in figure 1. In particular, a gap of order σ opens between the non-degenerate eigenvalue $\lambda^{\geqslant\sigma}_{j,g}$ and the essential spectrum of $H^{\geqslant\sigma}_{g,\theta}$. By Proposition 6.3, the same holds for $\widetilde{H}^{\geqslant\sigma}_{g,\theta}$ instead of $H^{\geqslant\sigma}_{g,\theta}$, with $\lambda_{j,g}$ replacing $\lambda^{\geqslant\sigma}_{j,g}$, since $|\lambda_{j,g} - \lambda^{\geqslant\sigma}_{j,g}| \leq \mathrm{C}g^2\sigma^{1+\mu}$ and we assumed that $g^2 \ll \sigma \ll 1$.

Let us begin to estimate $A(t,\overline{\theta}) - A(t,\theta)$ by considering the contribution of the regular part, $\widehat{R}^{\geqslant\sigma}_{g,\theta}(z)$, in $A(t,\theta)$. By applying Green's theorem, we find that

$$\begin{aligned} R(t,\theta) &:= \frac{1}{2i\pi}\int_{\mathbb{R}} e^{-itz} f(z)\left(\Psi_{j,\overline{\theta}}, \widehat{R}^{\geqslant\sigma}_{g,\theta}(z)\Psi_{j,\theta}\right) dz \\ &= \frac{1}{2i\pi}\int_{\Gamma(\gamma_1)} e^{-itz}\widetilde{f}(z)\left(\Psi^{\geqslant\sigma}_{j,\overline{\theta}}, \widehat{R}^{\geqslant\sigma}_{g,\theta}(z)\Psi^{\geqslant\sigma}_{j,\theta}\right) dz \\ &\quad + \frac{1}{2i\pi}\iint_{\mathrm{D}(\gamma_1)} e^{-itz}(\partial_{\overline{z}}\widetilde{f})(z)\left(\Psi^{\geqslant\sigma}_{j,\overline{\theta}}, \widehat{R}^{\geqslant\sigma}_{g,\theta}(z)\Psi^{\geqslant\sigma}_{j,\theta}\right) dz d\overline{z}, \end{aligned} \tag{3.20}$$

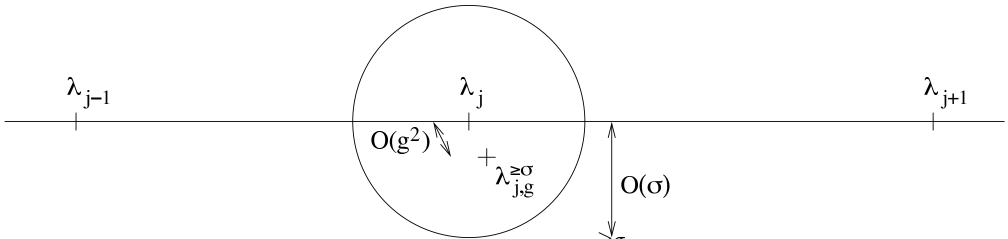

Figure 1: **Spectrum of $H^{\geqslant\sigma}_{g,\theta}$ near λ_j**

where $\Psi^{\geqslant\sigma}_{j,\theta} = \psi_{j,\theta} \otimes \Omega^{\geqslant\sigma}$, and $\Gamma(\gamma_1)$ and $\mathrm{D}(\gamma_1)$ denote respectively the curve and the domain pictured in figure 2, such that the interval I_0 strictly contains I.

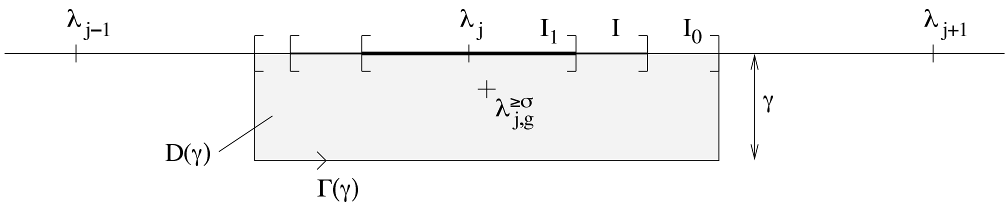

Figure 2: **Deformation of the path of integration**

By Proposition 4.1 and (3.8), the regular part $\widehat{R}^{\geqslant\sigma}_{g,\theta}(z)$ in (3.5) is analytic in $z \in \mathrm{D}(\gamma_1)$ and satisfies

$$\left\|\widehat{R}^{\geqslant\sigma}_{g,\theta}(z)\right\| \leq \frac{\mathrm{C}}{\mathrm{dist}(z, \sigma(\widetilde{H}^{\geqslant\sigma}_{g,\theta}) \setminus \{\lambda_{j,g}\})}, \tag{3.21}$$

where C is a positive constant. We also have from (3.9) that

$$P^{\geqslant\sigma}_{0,\theta}\widehat{R}^{\geqslant\sigma}_{g,\theta}(z)P^{\geqslant\sigma}_{0,\theta} = (P^{\geqslant\sigma}_{0,\theta} - P^{\geqslant\sigma}_{g,\theta})\widehat{R}^{\geqslant\sigma}_{g,\theta}(z)(P^{\geqslant\sigma}_{0,\theta} - P^{\geqslant\sigma}_{g,\theta}), \tag{3.22}$$

and from (3.10) that

$$P^{\geqslant\sigma}_{0,\theta} = P^{\geqslant\sigma}_{0,\theta}P^{\geqslant\sigma}_{g,\theta}P^{\geqslant\sigma}_{0,\theta} - (P^{\geqslant\sigma}_{g,\theta} - P^{\geqslant\sigma}_{0,\theta})(P^{\geqslant\sigma}_{g,\theta} - 1)(P^{\geqslant\sigma}_{g,\theta} - P^{\geqslant\sigma}_{0,\theta}), \tag{3.23}$$

and from Proposition 4.2, below, that

$$\|P^{\geqslant\sigma}_{g,\theta} - P^{\geqslant\sigma}_{0,\theta}\| \leq \mathrm{C}g\sigma^{-1/2}, \tag{3.24}$$

for some positive constant C. Thus, by (3.20) – (3.24), our assumptions on $\widetilde{f}$ and the fact that $|I_0| = O(\delta)$, we get

$$|R(t,\theta)| = O(\delta g^2\sigma^{-2}e^{-t\gamma_1}) + O(|\gamma_1/\delta|^n), \tag{3.25}$$

where $0 < \gamma_1 < \sigma\sin(\mathrm{Im}\theta)$, and any $n \in \mathbb{N}$. Similarly, for the contribution of the regular part $\widehat{R}^{\geqslant\sigma}_{g,\overline{\theta}}$ in $A(t,\overline{\theta})$, we have the following estimate

$$\left|R(t,\overline{\theta})\right| = O(\delta g^2\sigma^{-2}e^{-t\gamma_2}) + O(|\gamma_2/\delta|^n), \tag{3.26}$$

where $0 < \gamma_2 < \sin(\mathrm{Im}\theta)\sigma$, and any $n \in \mathbb{N}$.

Next, we estimate the singular part of $A(t,\overline{\theta}) - A(t,\theta)$. It is given by

$$S(t,\overline{\theta}) - S(t,\theta) := \frac{C_g^\sigma(\overline{\theta})}{2i\pi}\int_{\mathbb{R}} e^{-itz} f(z)\left(z - \overline{\lambda_{j,g}}\right)^{-1} dz - \frac{C_g^\sigma(\theta)}{2i\pi}\int_{\mathbb{R}} e^{-itz} f(z)\left(z - \lambda_{j,g}\right)^{-1} dz, \tag{3.27}$$

where we use the notation

$$C_g^\sigma(\theta) = \left(\Psi_j^{\geqslant\sigma}(\overline{\theta}), P_{g,\theta}^{\geqslant\sigma}\Psi_{j,\theta}^{\geqslant\sigma}\right). \tag{3.28}$$

By Proposition 4.2, we know that

$$C_g^\sigma(\theta) = 1 + O(g^2\sigma^{-1}). \tag{3.29}$$

We deform the path of integration as we did above, adding a circle $\mathcal{C}_\rho$ of radius ρ around $\lambda_{j,g}$. This yields

$$\begin{aligned} S(t,\overline{\theta}) - S(t,\theta) = & \frac{1}{2i\pi}\int_{\Gamma(\gamma_3)} e^{-itz}\widetilde{f}(z)\left[\frac{C_g^\sigma(\overline{\theta})}{z-\overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z-\lambda_{j,g}}\right] dz \\ & + \frac{1}{2i\pi}\int_{\mathcal{C}_\rho} e^{-itz}\widetilde{f}(z)\left[\frac{C_g^\sigma(\overline{\theta})}{z-\overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z-\lambda_{j,g}}\right] dz \\ & + \frac{1}{2i\pi}\iint_{\mathrm{D}(\gamma_3)\setminus\mathrm{D}_\rho} e^{-itz}(\partial_{\overline{z}}\widetilde{f})(z)\left[\frac{C_g^\sigma(\overline{\theta})}{z-\overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z-\lambda_{j,g}}\right] dz d\overline{z}, \end{aligned} \tag{3.30}$$

for all $\rho > 0$ sufficiently small, where D_ρ denotes the disc of radius ρ centered at $\lambda_{j,g}$, and $0 < \gamma_3 < \sin(\mathrm{Im}\theta)\sigma$. The first integral can be estimated by using arguments similar to those used to estimate the regular part, (3.29), and the fact that

$$\mathrm{Im}\lambda_{j,g} = O(g^2).$$

We then obtain that for $0 < g^2 < \sigma \ll 1$

$$\left|\frac{1}{2i\pi}\int_{\Gamma(\gamma_3)} e^{-itz}\widetilde{f}(z)\left[\frac{C_g^\sigma(\overline{\theta})}{z-\overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z-\lambda_{j,g}}\right] dz\right| = O(\delta g^2\sigma^{-2}e^{-t\gamma_3}). \tag{3.31}$$

Similarly, since $(\partial_{\overline{z}}\widetilde{f}) = 0$ on $\{z|\ \mathrm{Re}(z) \in \mathrm{I}_1\}$, we see that the third integral in the r.h.s. of (3.30) is independent of ρ, for ρ sufficiently small, and that for any $n \in \mathbb{N}$

$$\left|\frac{1}{2i\pi}\iint_{\mathrm{D}(\gamma_3)\setminus\mathrm{D}_\rho} e^{-itz}(\partial_{\overline{z}}\widetilde{f})(z)\left[\frac{C_g^\sigma(\overline{\theta})}{z-\overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z-\lambda_{j,g}}\right] dz\right| = O(|\gamma_3/\delta|^n). \tag{3.32}$$

It remains to estimate the second integral on the right hand side of (3.30). Taking the limit as $\rho \to 0$ leads to the "residue" $C_g^\sigma(\theta)e^{-it\lambda_{j,g}}\widetilde{f}(\lambda_{j,g})$. Since, by construction, $\widetilde{f} = 1$ on $\{z|\ \mathrm{Re}(z) \in \mathrm{I}_1\}$, we get

$$\lim_{\rho\to 0}\frac{1}{2i\pi}\int_{\mathcal{C}_\rho} e^{-itz}\widetilde{f}(z)\left[\frac{C_g^\sigma(\overline{\theta})}{z-\overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z-\lambda_{j,g}}\right] dz = C_g^\sigma(\theta)e^{-it\lambda_{j,g}}. \tag{3.33}$$

The claim of the lemma follows from (3.25) – (3.33). □

Lemma 3.4 *Assume that the infrared cut-off parameter σ is chosen such that $g^2 \ll \sigma \ll 1$. Then, for all times $t \geq 0$, we have that*

$$\big|B(t,\overline{\theta}) - B(t,\theta)\big| = O(\delta g^{-2}\sigma^{\frac{1}{2}+\mu}), \tag{3.34}$$

where $B(t,\theta)$ is defined in (3.13).

Proof. Recall that

$$B(t,\theta) = \sum_{n\geq 1} B^n(t,\theta), \tag{3.35}$$

where

$$B^n(t,\theta) = \frac{1}{2i\pi}\int_{\mathbb{R}} e^{-itz} f(z) \left(\Psi_j(\overline{\theta}), \widetilde{R}^{\sigma}_{g,\theta}(z)\left(-\widetilde{W}^{\leqslant\sigma}_{g,\theta}\widetilde{R}^{\sigma}_{g,\theta}(z)\right)^n \Psi_j(\theta)\right) dz. \tag{3.36}$$

It follows from (3.36) and Lemma 4.4 that[3]

$$|B^n(t,\theta)| = O(\delta g^{-2}\sigma^{n(\frac{1}{2}+\mu)}), \tag{3.37}$$

uniformly in $t \geq 0$. Together with (3.35) and the assumption on σ and g, it follows that

$$|B(t,\theta)| = O(\delta g^{-2}\sigma^{\frac{1}{2}+\mu}), \tag{3.38}$$

uniformly in t. One can similarly show that

$$|B(t,\overline{\theta})| = O(\delta g^{-2}\sigma^{\frac{1}{2}+\mu}), \tag{3.39}$$

and hence the claim of the lemma follows. □

Proof of Proposition 3.1. It follows from Lemmata 3.2, 3.3 and 3.4 that for $t \geq 0$

$$\big(\Psi_j, e^{-itH_g} f(H_g)\Psi_j\big) = e^{-it\lambda_{j,g}} + O(\delta g^2\sigma^{-2}) + O(\delta g^{-2}\sigma^{1/2+\mu}) + O(g^2\sigma^{-1}). \tag{3.40}$$

Let $\delta = C\sigma$, for some $C > 1$. We optimize the estimate on the error term by choosing

$$\sigma = g^{2-\frac{2+4\mu}{5+2\mu}}, \tag{3.41}$$

and hence the claim of the proposition is proven. □

Proof of Theorem 1.1. Proposition 3.1 implies that, for $t = 0$,

$$(\Psi_j, (1 - f(H_g))\Psi_j) = \|\sqrt{1 - f(H_g)}\Psi_j\|^2 = O(g^{\frac{2+4\mu}{5+2\mu}}), \tag{3.42}$$

which, together with the boundedness of the unitary operator e^{-itH_g} and Proposition 3.1, for arbitrary $t > 0$, yields

$$\begin{aligned}(\Psi_j, e^{-itH_g}\Psi_j) &= (\Psi_j, e^{-itH_g}(1 - f(H_g) + f(H_g))\Psi_j)\\ &= (\Psi_j, e^{-itH_g} f(H_g)\Psi_j) + O(\|\sqrt{1 - f(H_g)}\Psi_j\|^2)\\ &= e^{-it\lambda_{j,g}} + O(g^{\frac{2+4\mu}{5+2\mu}}).\end{aligned}$$

□

[3] Estimate (3.37) can be improved if one uses instead of Ψ_j a state that is a better approximation to the resonance eigenstate.

4 The Hamiltonian $H^{\sigma}_{g,\theta}$

In this section, we study the operator $H^{\sigma}_{g,\theta}$ used in the previous section as an approximation of $H_{g,\theta}$. We use the Feshbach-Schur map[4], [4, 5], defined for a projection P and a closed operator H whose domain is contained in $\mathrm{Ran}(P)$, by

$$\mathcal{F}_P(H) = PHP - PH\overline{P}\left[\overline{P}H\overline{P}\right]^{-1}\overline{P}HP, \tag{4.1}$$

where $\overline{P} = 1 - P$. Note that the domain of $\mathcal{F}_P$ consists of operators H such that

$$\left[\overline{P}H\overline{P}\right]^{-1}|_{\mathrm{Ran}(\overline{P})}, \quad PH\overline{P}\left[\overline{P}H\overline{P}\right]^{-1}|_{\mathrm{Ran}(\overline{P})}, \quad \left[\overline{P}H\overline{P}\right]^{-1}\overline{P}HP \tag{4.2}$$

extend to bounded operators. We begin with the following proposition.

Proposition 4.1 *Suppose $0 < g^2 \ll \sigma \ll 1$. Then, for $\theta \in D(0,\theta_0)$ such that $\mathrm{Im}\theta \neq 0$ and $\sigma < \mathrm{d}_j \sin|\mathrm{Im}\theta|$, the spectrum of $H^{\geqslant\sigma}_{g,\theta}$ in the disc $D(\lambda_j, \sigma/2)$ consists of a single eigenvalue*

$$\sigma(H^{\geqslant\sigma}_{g,\theta}) \cap D(\lambda_j, \sigma/2) = \{\lambda^{\geqslant\sigma}_{j,g}\}. \tag{4.3}$$

Furthermore, there exists $\varepsilon > 0$ such that, for all z in $D(\lambda_j, \sigma/3)$ such that $\left|z - \lambda^{\geqslant\sigma}_{j,g}\right| \gg g^{2+\varepsilon}$,

$$\left\|R^{\geqslant\sigma}_{g,\theta}(z)\right\| \leq \frac{\mathrm{C}}{\mathrm{dist}(z, \sigma(H^{\geqslant\sigma}_{g,\theta}))}, \tag{4.4}$$

for some positive constant C.

Proof. Let $P_\theta := P^{\geqslant\sigma}_{0,\theta} = P_{p,j,\theta} \otimes P^{\geqslant\sigma}_{\Omega}$ and $\overline{P}_\theta := 1 - P_\theta$. For $\sigma < \mathrm{d}_j \sin|\mathrm{Im}\theta|$ and $z \in D(\lambda_j, \sigma/2)$, one can show that, for any $n \geq 1$,

$$\left\|\left[\overline{P}_\theta H^{\geqslant\sigma}_{0,\theta} - z\right]^{-1}\left(-W^{\geqslant\sigma}_{g,\theta}\left[\overline{P}_\theta H^{\geqslant\sigma}_{0,\theta} - z\right]^{-1}\right)^n\right\| \leq \mathrm{C}_1\sigma^{-1}\left(\mathrm{C}_2 g\sigma^{-1/2}\right)^n, \tag{4.5}$$

where C_1, C_2 are positive constant. Hence for $g\sigma^{-1/2} \ll 1$ and any z in $D(\lambda_j, \sigma/2)$, the operator $\overline{P}_\theta H^{\geqslant\sigma}_{g,\theta}\overline{P}_\theta - z$ is invertible and its inverse is given by the convergent Neumann series as

$$\left[\overline{P}_\theta H^{\geqslant\sigma}_{g,\theta}\overline{P}_\theta - z\right]^{-1} = \left[\overline{P}_\theta H^{\geqslant\sigma}_{0,\theta} - z\right]^{-1}\sum_{n\geq 0}\left(-W^{\geqslant\sigma}_{g,\theta}\left[\overline{P}_\theta H^{\geqslant\sigma}_{0,\theta} - z\right]^{-1}\right)^n. \tag{4.6}$$

This implies that the operator $H^{\geqslant\sigma}_{g,\theta} - z$ is in the domain of $\mathcal{F}_{P_\theta}$. Moreover, (4.5) and (4.6) lead to the estimates

$$\left\|(H_f + 1)^n\left[\overline{P}_\theta H^{\geqslant\sigma}_{g,\theta}\overline{P}_\theta - z\right]^{-1}\right\| \leq \mathrm{C}\sigma^{-1}, \quad n = 0, 1, \tag{4.7}$$

for some positive constant C.

[4]In [4, 5, 12] this map is called the Feshbach map. As was pointed out to us by F. Klopp and B. Simon, the invertibility procedure at the heart of this map was introduced by I. Schur in 1917; a similar approach was developed in an independent work of H. Feshbach on the theory of nuclear reactions in 1958, see [13] for further extensions and historical remarks.

Note that our choice of P_θ yields $P_\theta W_{g,\theta}^{\geqslant\sigma} P_\theta = 0$. Therefore

$$\mathcal{F}_{P_\theta}(H_{g,\theta}^{\geqslant\sigma} - z) = (\lambda_j - z) P_\theta - P_\theta W_{g,\theta}^{\geqslant\sigma} \overline{P}_\theta \left[\overline{P}_\theta H_{g,\theta}^{\geqslant\sigma} \overline{P}_\theta - z \right]^{-1} \overline{P}_\theta W_{g,\theta}^{\geqslant\sigma} P_\theta. \tag{4.8}$$

The non-degeneracy of λ_j implies that $\mathcal{F}_{P_\theta}(H_{g,\theta}^{\geqslant\sigma} - z)$ can be written as $[\lambda_j - z + a(z)]P_\theta$, where $a(z)$ is a function from $D(\lambda_j, \sigma/2) \to \mathbb{C}$. Following [4, 6] (see also Proposition 4.3 below), we have

$$a(z) = g^2 Z_{j,\theta} + O(g^{2+\varepsilon}) \tag{4.9}$$

for some $\varepsilon > 0$, where $Z_{j,\theta} := Z_{j,\theta}^{\mathrm{od}} + Z_{j,\theta}^{\mathrm{d}}$ with

$$Z_{j,\theta}^{\mathrm{od}} = \int_{\mathbb{R}^3} \mathcal{U}_\theta P_{p,j} \overline{G_x}(k) \overline{P}_{p,j} \left[H_p - \lambda_j + \omega(k) - i0 \right]^{-1} \overline{P}_{p,j} G_x(k) P_{p,j} \mathcal{U}_\theta^{-1} dk, \tag{4.10}$$

$$Z_{j,\theta}^{\mathrm{d}} = \int_{\mathbb{R}^3} \mathcal{U}_\theta P_{p,j} \overline{G_x}(k) P_{p,j} G_x(k) P_{p,j} \mathcal{U}_\theta^{-1} \frac{dk}{\omega(k)}. \tag{4.11}$$

Using the Leibniz rule and the fact that

$$\frac{d}{dz} \left[\overline{P}_\theta H_{0,\theta}^{\geqslant\sigma} - z \right]^{-1} = \left[\overline{P}_\theta H_{0,\theta}^{\geqslant\sigma} - z \right]^{-2}, \tag{4.12}$$

one can prove, by differentiating (4.8) with respect to z, that $z \mapsto b(z) := \lambda_j - z + a(z)$ is an analytic function on $D(\lambda_j, \sigma/2)$, and that $|db(z)/dz - 1| < 1$, provided that $g^2\sigma^{-1}$ is sufficiently small. This implies that b is a bijection on $D(\lambda_j, \sigma/2)$.

The isospectrality of the Feshbach map (see [4, 5]) tells us that

$$z \in \sigma(H_{g,\theta}^{\geqslant\sigma}) \iff 0 \in \mathcal{F}_{P_\theta}(H_{g,\theta}^{\geqslant\sigma} - z) \iff b(z) = 0.$$

On the other hand, it follows from the usual perturbation theory, applied to the isolated non-degenerate eigenvalue λ_j of $H_{0,\theta}^{\geqslant\sigma}$, that the spectrum of $H_{g,\theta}^{\geqslant\sigma}$ is not empty in $D(\lambda_j, \sigma/2)$, for g sufficiently small. Hence there exists a unique $\lambda_{j,g}^{\geqslant\sigma}$ in $D(\lambda_j, \sigma/2)$ such that $b(\lambda_{j,g}^{\geqslant\sigma}) = 0$, that is

$$\sigma(H_{g,\theta}^{\geqslant\sigma}) \cap D(\lambda_j, \frac{\sigma}{2}) = \{\lambda_{j,g}^{\geqslant\sigma}\}. \tag{4.13}$$

To prove (4.4), we use the following identity (see [4]):

$$\begin{aligned} \left[H_{g,\theta}^{\geqslant\sigma} - z \right]^{-1} &= \left[P_\theta - \left[\overline{P}_\theta \left(H_{g,\theta}^{\geqslant\sigma} - z \right) \overline{P}_\theta \right]^{-1} \overline{P}_\theta W_{g,\theta}^{\geqslant\sigma} P_\theta \right] \left[\mathcal{F}_{P_\theta}(H_{g,\theta}^{\geqslant\sigma} - z) \right]^{-1} \\ &\quad \times \left[P_\theta - P_\theta W_{g,\theta}^{\geqslant\sigma} \overline{P}_\theta \left[\overline{P}_\theta \left(H_{g,\theta}^{\geqslant\sigma} - z \right) \overline{P}_\theta \right]^{-1} \right] \\ &\quad + \left[\overline{P}_\theta \left(H_{g,\theta}^{\geqslant\sigma} - z \right) \overline{P}_\theta \right]^{-1} \overline{P}_\theta, \end{aligned} \tag{4.14}$$

which holds for z in $\rho(H_{g,\theta}^{\geqslant\sigma}) \cap D(\lambda_j, \sigma/2)$. The simple form of $\mathcal{F}_{P_\theta}(H_{g,\theta}^{\geqslant\sigma} - z)$, (4.7) and the fact that $|a(z) - a(\lambda_{j,g}^{\geqslant\sigma})| = O(g^{2+\varepsilon})$ by (4.9) lead to

$$\left\| \left[H_{g,\theta}^{\geqslant\sigma} - z \right]^{-1} \right\| \leq \mathrm{C}_1 \left(\frac{1 + g^2\sigma^{-1}}{|z - \lambda_{j,g}^{\geqslant\sigma}| - \mathrm{C}_2 g^{2+\varepsilon}} + \sigma^{-1} \right), \tag{4.15}$$

for some positive constants C_1, C_2. Hence the proposition is proven for z in $D(\lambda_j, \sigma/3)$ such that $|z - \lambda_{j,g}^{\geqslant\sigma}| \gg g^{2+\varepsilon}$. $\Box$

Recall that, for $g \geq 0$, $P_{g,\theta}^{\geqslant\sigma}$ denotes the projection onto the eigenspace associated with the eigenvalue $\lambda_{j,g}^{\geqslant\sigma}$ of $H_{g,\theta}^{\geqslant\sigma}$.

Proposition 4.2 *Let g, σ as in Proposition 4.1 and choose $\theta \in D(0, \theta_0)$ such that $\mathrm{Im}\theta \neq 0$. Then, for g small enough,*

$$\left\| P_{g,\theta}^{\geqslant\sigma} - P_{0,\theta}^{\geqslant\sigma} \right\| \leq C_0 g \sigma^{-1/2}, \tag{4.16}$$

where C_0 is a positive constant.

Proof. Let $\mathcal{C}_j$ denote a circle centered at λ_j, with radius $\sigma/3$, so that $\mathcal{C}_j \subset \rho(H_{g,\theta}^{\geqslant\sigma})$. Since we have assumed $g^2 \ll \sigma$, for g sufficiently small, $\mathcal{C}_j$ contains both $\lambda_{j,g}^{\geqslant\sigma}$ and λ_j. Thus,

$$P_{g,\theta}^{\geqslant\sigma} - P_{0,\theta}^{\geqslant\sigma} = \frac{1}{2\pi i} \oint_{\mathcal{C}_j} \left[R_{g,\theta}^{\geqslant\sigma}(z) - R_{0,\theta}^{\geqslant\sigma}(z) \right] dz. \tag{4.17}$$

We expand $R_{g,\theta}^{\geqslant\sigma}(z)$ into a Neumann series

$$R_{g,\theta}^{\geqslant\sigma}(z) = R_{0,\theta}^{\geqslant\sigma}(z) \sum_{n\geq 0} \left(-W_{g,\theta}^{\geqslant\sigma} R_{0,\theta}^{\geqslant\sigma}(z) \right)^n. \tag{4.18}$$

One can show by following the method of [6] that, for $n = 0, 1/2, 1$,

$$\|(H_f^{\geqslant\sigma})^n R_{0,\theta}^{\geqslant\sigma}\| = O(\sigma^{-1+n}). \tag{4.19}$$

Hence, using that $a(G_{x,\theta}^{\geqslant\sigma}) P_\Omega^{\geqslant\sigma} = 0$ and $\|a(G_{x,\theta}^{\leqslant\sigma})(H_f^{\geqslant\sigma})^{-1/2} \overline{P}_\Omega^{\geqslant\sigma}\| = O(1)$, we obtain that for all $n \geq 1$,

$$\left\| R_{0,\theta}^{\geqslant\sigma}(z) \left(-W_{g,\theta}^{\geqslant\sigma} R_{0,\theta}^{\geqslant\sigma}(z) \right)^n \right\| \leq \frac{C_1}{\sigma} \left(C_2 g \sigma^{-1/2} \right)^n, \tag{4.20}$$

where C_1 and C_2 denote positive constants. Inserting this in (4.17) and using the fact that the radius of $\mathcal{C}_j$ is equal to $\sigma/3$, we obtain

$$\left\| P_{g,\theta}^{\geqslant\sigma} - P_{0,\theta}^{\geqslant\sigma} \right\| = \frac{1}{2\pi} \left\| \oint_{\mathcal{C}_j} R_{0,\theta}^{\geqslant\sigma}(z) \sum_{n\geq 1} \left(-W_{g,\theta}^{\geqslant\sigma} R_{0,\theta}^{\geqslant\sigma}(z) \right)^n dz \right\| \leq C_0 g \sigma^{-1/2}, \tag{4.21}$$

provided that $g\sigma^{-1/2}$ is sufficiently small. Hence the proposition is proven. $\Box$

Using a renormalization group analysis, we will prove in Proposition 6.3 below the following estimate of the difference between the eigenvalues $\lambda_{j,g}$ and $\lambda_{j,g}^{\geqslant\sigma}$ of $H_{g,\theta}$ and $H_{g,\theta}^{\sigma}$:

$$\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma} = O(g^2 \sigma^{1+\mu}),$$

for any $\sigma > 0$. Here we prove a weaker estimate, which holds only for $\sigma \gg g^2$, but which does not require the use of a renormalization group analysis. Besides, it is sufficient to obtain the statement of Theorem 1.1, with the slightly weaker error term $O(g^{\frac{1+2\mu}{4+2\mu}})$ for the Nelson model, and $O(g^{-1/4})$ for the QED one.

Proposition 4.3 *Suppose* $0 < g^2 \ll \sigma < g^{\frac{3}{2+\mu}}$. *Then*

$$\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma} = O\left(g^{4-\frac{3}{2+\mu}}\right). \tag{4.22}$$

Proof. For g and σ small enough, we choose $\theta \in D(0,\theta_0)$, $\mathrm{Im}(\theta) \neq 0$, such that $0 < g^2 \ll \sigma < \mathrm{d}_j \sin|\mathrm{Im}\theta| < 1$. For ρ such that $g^2 \ll \rho < \mathrm{d}_j \sin|\mathrm{Im}\theta|$, let $P_\theta := P_{p,j,\theta} \otimes \mathbf{1}_{H_f<\rho}$. Following [4, 6], $\lambda_{j,g}$ satisfies

$$\left|\lambda_{j,g} - \lambda_j - g^2 Z_{j,\theta}\right| \leq \sum_{i=1}^{5} \|\mathrm{Rem}_i\|, \tag{4.23}$$

where $Z_{j,\theta} := Z_{j,\theta}^{\mathrm{od}} + Z_{j,\theta}^{\mathrm{d}}$, with $Z_{j,\theta}^{\mathrm{od}}$ and $Z_{j,\theta}^{\mathrm{d}}$ given by (4.10)-(4.11), and

$$\mathrm{Rem}_1 = P_\theta W_{g,\theta} P_\theta, \tag{4.24}$$

$$\mathrm{Rem}_2 = P_\theta W_{g,\theta} \overline{P}_\theta \left[\overline{P}_\theta H_{0,\theta} - \lambda_{j,g}\right]^{-1} \overline{P}_\theta W_{g,\theta} P_\theta - g^2 Q_\theta \tag{4.25}$$

$$\mathrm{Rem}_3 = g^2 \left[Q_\theta - Z_{j,\theta}\right], \tag{4.26}$$

$$\mathrm{Rem}_4 = P_\theta W_{g,\theta} \left(\overline{P}_\theta \left[\overline{P}_\theta H_{0,\theta} - \lambda_{j,g}\right]^{-1} \overline{P}_\theta W_{g,\theta}\right)^2 P_\theta, \tag{4.27}$$

$$\mathrm{Rem}_5 = P_\theta W_{g,\theta} \sum_{n\geq 3} \left(\overline{P}_\theta \left[\overline{P}_\theta H_{0,\theta} - \lambda_{j,g}\right]^{-1} \overline{P}_\theta W_{g,\theta}\right)^n P_\theta. \tag{4.28}$$

Here we have set

$$Q_\theta = \int_{\mathbb{R}^3} P_\theta \overline{G_{x,\theta}}(k) \left[\frac{\overline{P}_\theta(\omega(k))}{H_{0,\theta} + e^{-\theta}\omega(k) - \lambda_{j,g}}\right] G_{x,\theta}(k) P_\theta dk, \tag{4.29}$$

where $\overline{P}_\theta(\omega(k)) := 1 - P_\theta(\omega(k))$, and $P_\theta(\omega(k)) := P_{p,j,\theta} \otimes \mathbf{1}_{H_f+\omega(k)<\rho}$. Using the expression (2.1) of $G_{x,\theta}$ and estimates similar to [4, Lemmas IV.6-IV.12] or [6, Lemma 3.16], we claim that

$$\|\mathrm{Rem}_1\| = O\left(g\rho^{1+\mu}\right), \quad \|\mathrm{Rem}_2\| = O\left(g^2\rho^{1+\mu}\right), \quad \|\mathrm{Rem}_3\| = O\left(g^2\rho\right). \tag{4.30}$$

The first bound in (4.30) easily follows from

$$\int_{\mathbb{R}^3} \|P_\theta \overline{G_{x,\theta}(k)} \otimes a^*(k)\| dk = \int_{\mathbb{R}^3} \|G_{x,\theta}(k) \otimes a(k) P_\theta\| dk \leq \mathrm{C}\rho^{1+\mu}. \tag{4.31}$$

The second one follows from normal-ordering (4.25) and using again (4.31). Finally, the last bound in (4.30) follows from computing the difference in (4.26) and using the estimate

$$\left\|\frac{\overline{P}_\theta(\omega(k))}{H_{0,\theta} + e^{-\theta}\omega(k) - \lambda_{j,g}}\right\| \leq \frac{\mathrm{C}_1}{(\mathrm{d}_j \sin|\mathrm{Im}\theta|)\omega(k)}, \tag{4.32}$$

for some positive constant C_1. Now it is proved in [4, 6] that $\|\mathrm{Rem}_4 + \mathrm{Rem}_5\| = O\left(g^3\rho^{-1/2}\right)$. Let us estimate these terms more precisely: we claim that

$$\|\mathrm{Rem}_4\| = O\left(g^3\rho^{\mu}\right), \quad \|\mathrm{Rem}_5\| = O\left(g^4\rho^{-1}\right). \tag{4.33}$$

To prove the first bound in (4.33), we decompose $W_{g,\theta}$ into $W_{g,\theta} = g(a^*(G_{x,\theta}) + a(G_{x,\theta}))$ and estimate each term separately by normal ordering. For instance, let us compute

$$
\begin{aligned}
&P_\theta a(G_{x,\theta}) \frac{\overline{P}_\theta}{H_{0,\theta} - \lambda_{j,g}} a^*(G_{x,\theta}) \frac{\overline{P}_\theta}{H_{0,\theta} - \lambda_{j,g}} a(G_{x,\theta}) P_\theta \\
&= P_\theta \int_{\mathbb{R}^9} G_{x,\theta}(k_1) \otimes a(k_1) \frac{\overline{P}_\theta}{H_{0,\theta} - \lambda_{j,g}} \overline{G}_{x,\theta}(k_2) \otimes a^*(k_2) \frac{\overline{P}_\theta}{H_{0,\theta} - \lambda_{j,g}} \\
&\qquad G_{x,\theta}(k_3) \otimes a(k_3) P_\theta dk_1 dk_2 dk_3.
\end{aligned}
\tag{4.34}
$$

It follows from a pull-through formula and the canonical commutation rules that the "worst" term we have to estimate from the rhs of (4.34) is

$$
\begin{aligned}
\mathrm{T}(\rho, \theta) := P_\theta \int_{\mathbb{R}^6} & G_{x,\theta}(k_1) \otimes I \frac{\overline{P}_\theta(\omega(k_1))}{H_{0,\theta} + e^{-\theta}\omega(k_1) - \lambda_{j,g}} \overline{G}_{x,\theta}(k_1) \\
& \otimes I \frac{\overline{P}_\theta}{H_{0,\theta} - \lambda_{j,g}} G_{x,\theta}(k_3) \otimes a(k_3) P_\theta dk_1 dk_3.
\end{aligned}
\tag{4.35}
$$

One can see that

$$
\left\| \frac{\overline{P}_\theta}{H_{0,\theta} - \lambda_{j,g}} \right\| \leq \frac{\mathrm{C}_1}{(\mathrm{d}_j \sin |\mathrm{Im}\theta|)\rho}, \tag{4.36}
$$

for some positive constant C_1. Inserting this together with (4.32) into (4.35), we get

$$
\begin{aligned}
\|\mathrm{T}(\rho, \theta)\| &\leq \frac{\mathrm{C}_1^2}{(\mathrm{d}_j \sin |\mathrm{Im}\theta|)^2} \rho^{-1} \int_{\mathbb{R}^3} \frac{|G_{x,\theta}(k_1)|^2}{\omega(k_1)} dk_1 \int_{\mathbb{R}^3} \|G_{x,\theta}(k_3) \otimes a(k_3) P_\theta\| dk_3 \\
&\leq \frac{\mathrm{C}_2}{(\mathrm{d}_j \sin |\mathrm{Im}\theta|)^2} \rho^\mu,
\end{aligned}
\tag{4.37}
$$

where C_2 is a positive constant. Since the other terms could be estimated in the same way, the first bound in (4.33) follows; the second bound in (4.33) can be obtained by using similar computations (see also [6, Lemma 3.16]).

For $\rho > \sigma$ the eigenvalue $\lambda_{j,g}^{\geqslant\sigma}$ of $H_{g,\theta}^{\geqslant\sigma}$ is given by the formulas (4.23)–(4.29), except that $W_{g,\theta}$ and $G_{x,\theta}(k)$ are replaced respectively by $W_{g,\theta}^{\geqslant\sigma}$ and $G_{x,\theta}^{\geqslant\sigma}(k)$. For the terms analogous to $Z_{j,\theta}^{\mathrm{od}}$ and $Z_{j,\theta}^{\mathrm{d}}$ we have by a straightforward computations that

$$
\int_{|k|\leq\sigma} \mathcal{U}_\theta P_{p,j} \overline{G_x}(k) \overline{P}_{p,j} \left[H_p - \lambda_j + \omega(k) - i0\right]^{-1} \overline{P}_{p,j} G_x(k) P_{p,j} \mathcal{U}_\theta^{-1} dk = 0, \tag{4.38}
$$

$$
\int_{|k|\leq\sigma} \mathcal{U}_\theta P_{p,j} \overline{G_x}(k) P_{p,j} G_x(k) P_{p,j} \mathcal{U}_\theta^{-1} \frac{dk}{\omega(k)} = O\left(\sigma^{1+2\mu}\right), \tag{4.39}
$$

where in (4.38) we used the fact that $\sigma < \mathrm{d}_j$. Hence, with the obvious notation, $Z_{j,\theta} = Z_{j,\theta}^{\geqslant\sigma} + O(\sigma^{1+2\mu})$. Furthermore, Eqns. (4.30)–(4.33) still hold for $\lambda_{j,g}^{\geqslant\sigma}$. Hence remembering the assumptions $\sigma < \rho$, $g^2 \ll \rho$ we obtain

$$
\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma} = O\left(g\rho^{1+\mu}\right) + O\left(g^2\rho\right) + O\left(g^4\rho^{-1}\right). \tag{4.40}
$$

Optimizing with respect to ρ leads to the claim of the proposition. $\square$

The following lemma was used in the proof of Lemmata 3.2 and 3.4.

Lemma 4.4 *Let θ in $D(0,\theta_0)$, $\mathrm{Im}\theta > 0$ and let g, σ be such that $0 < g^2 \ll \sigma \ll 1$. Then for all $z \in \mathrm{I}$ and $n \geq 1$, we have the estimate:*

$$\left\| \widetilde{R}^{\sigma}_{g,\theta}(z) \left(\widetilde{W}^{\leqslant\sigma}_{g,\theta} \widetilde{R}^{\sigma}_{g,\theta}(z) \right)^n \right\| \leq \mathrm{C}_1 g^{-2} \left(\mathrm{C}_2 \sigma^{1/2+\mu} \right)^n, \tag{4.41}$$

where $\mathrm{C}_1, \mathrm{C}_2$ are positive constants.

Proof. Recall that

$$\widetilde{W}^{\leqslant\sigma}_{g,\theta} = g a^*(G^{\leqslant\sigma}_{x,\theta}) + g a(G^{\leqslant\sigma}_{x,\theta}) + \left(\lambda_{j,g} - \lambda^{\geqslant\sigma}_{j,g} \right). \tag{4.42}$$

From the spectral representation (3.18) and the decomposition (3.5), we can write

$$\widetilde{R}^{\sigma}_{g,\theta}(z) = \widehat{R}^{\sigma}_{g,\theta}(z) + (P^{\geqslant\sigma}_{g,\theta} \otimes I) \left[z - \lambda^{\geqslant\sigma}_{j,g} - e^{-\theta} H^{\leqslant\sigma}_f \right]^{-1}, \tag{4.43}$$

where $\widehat{R}^{\sigma}_{g,\theta}(z) := (I - (P^{\geqslant\sigma}_{g,\theta} \otimes I)) \widetilde{R}^{\sigma}_{g,\theta}(z)$. With the definition (3.7), we have

$$(H^{\leqslant\sigma}_f)^n \widehat{R}^{\sigma}_{g,\theta}(z) = \int_{\sigma(H^{\leqslant\sigma}_f)} \omega^n \widehat{R}^{\geqslant\sigma}_{g,\theta}(z - e^{-\theta}\omega) \otimes dE_{H^{\leqslant\sigma}_f}(\omega). \tag{4.44}$$

It follows from Proposition 4.1 that for all z in I and for $n = 0, 1/2, 1$

$$\left\| (H^{\leqslant\sigma}_f)^n \widehat{R}^{\sigma}_{g,\theta}(z) \right\| = O(\sigma^{-1+n}). \tag{4.45}$$

Besides, for $n = 0, 1/2, 1$

$$\left\| (H^{\leqslant\sigma}_f)^n \left[z - \lambda^{\geqslant\sigma}_{j,g} - e^{-\theta} H^{\leqslant\sigma}_f \right]^{-1} \right\| \leq \left| \mathrm{Im}(\lambda^{\geqslant\sigma}_{j,g}) \right|^{-1+n} = O(g^{-2(1-n)}), \tag{4.46}$$

provided that Fermi's Golden Rule holds. From $a(G^{\leqslant\sigma}_{x,\theta}) P^{\leqslant\sigma}_{\Omega} = 0$ and $\| a(G^{\leqslant\sigma}_{x,\theta}) (H^{\leqslant\sigma}_f)^{-1/2} \overline{P}^{\leqslant\sigma}_{\Omega} \| = O(\sigma^{1/2+\mu})$ (where $P^{\leqslant\sigma}_{\Omega}$ denotes the projection onto the vacuum in $\mathcal{F}^{\leqslant\sigma}_s$), we get

$$\left\| a(G^{\leqslant\sigma}_{x,\theta}) \left[z - \lambda^{\geqslant\sigma}_{j,g} - e^{-\theta} H^{\leqslant\sigma}_f \right]^{-1} \right\| = O(g^{-1} \sigma^{1/2+\mu}). \tag{4.47}$$

Similarly (4.45) leads to

$$\left\| a(G^{\leqslant\sigma}_{x,\theta}) \widehat{R}^{\sigma}_{g,\theta}(z) \right\| = O(\sigma^{\mu}). \tag{4.48}$$

The claim of the lemma then follows from (4.42) – (4.48), the assumption $g^2 \ll \sigma < g^{\frac{3}{2+\mu}}$, and Proposition 6.3. $\square$

5 Extension to non-relativistic QED

Now we extend the analysis above to the standard Hamiltonian of non-relativistic QED introduced in (1.2), Section 1. Let now $H_{g,\theta}$ be the dilatation deformation of the Hamiltonian H_g^{SM} defined in (1.10). We keep the notation of Sections 2 - 4.

The results and proofs of Sections 3 - 4 go through without a change except for the proof of Lemma 3.4. In the non-relativistic QED case, $W_{g,\theta}$ is given by

$$W_{g,\theta} = ge^{-\theta} p \cdot A_\theta(x) + \frac{g^2}{2} A_\theta(x) \cdot A_\theta(x) - \frac{g^2}{2}\Lambda, \tag{5.1}$$

where we used the notation $p \cdot A_\theta(x) := -i \sum_{j=1}^N \frac{1}{m_j} \nabla_j \cdot \mathcal{U}_\theta A(x_j) \mathcal{U}_\theta^{-1}$, and similarly for $A_\theta(x) \cdot A_\theta(x)$. The quantized vector potential $A(x_j)$ is given by (1.3), and the constant Λ is given by $\Lambda := \frac{1}{(2\pi)^3}(\sum_j \frac{1}{m_j}) \int \chi(k)^2/|k| d^3k$. Here we have

$$W_{g,\theta}^{\leqslant\sigma} = ge^{-\theta} p \cdot A_\theta^{\leqslant\sigma}(x) + g^2 A_\theta^{\leqslant\sigma}(x) \cdot A_\theta^{\geqslant\sigma}(x) + \frac{g^2}{2} A_\theta^{\leqslant\sigma}(x) \cdot A_\theta^{\leqslant\sigma}(x) - \frac{g^2}{2}\Lambda^{\leqslant\sigma}, \tag{5.2}$$

where $\Lambda^{\leqslant\sigma} := \frac{1}{(2\pi)^3}(\sum_j \frac{1}{m_j}) \int_{|k|\leq\sigma} \chi(k)^2/|k| d^3k$. Hence the QED Hamiltonian satisfies the condition similar to (2.7) with $\mu = 0$. We show now how to overcome this difficulty (a different way to proceed is to use the Pauli-Fierz transform [4, 6, 7]).

In our sketch of the proof of Lemma 3.4, we begin with the most singular term, B^1, of the expansion (3.35) in Section 3. Thus we have to bound the term $\widetilde{R}_{g,\theta}^\sigma \widetilde{W}_{g,\theta}^{\leqslant\sigma} \widetilde{R}_{g,\theta}^\sigma$. The part of $\widetilde{W}_{g,\theta}^{\leqslant\sigma}$ involving the difference of the eigenvalues is estimated in the same way as before. Namely, using that $\|\widetilde{R}_{g,\theta}^\sigma\| = O(g^{-2})$ and that, by Proposition 6.3, $|\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}| = O(g^2\sigma)$, we obtain that

$$\left\| \widetilde{R}_{g,\theta}^\sigma \left(\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma} \right) \widetilde{R}_{g,\theta}^\sigma \right\| = O(\sigma g^{-2}). \tag{5.3}$$

Now we estimate the remaining part $\widetilde{R}_{g,\theta}^\sigma W_{g,\theta}^{\leqslant\sigma} \widetilde{R}_{g,\theta}^\sigma$ of $\widetilde{R}_{g,\theta}^\sigma \widetilde{W}_{g,\theta}^{\leqslant\sigma} \widetilde{R}_{g,\theta}^\sigma$. Using the relation $e^{-\theta} p_j = m_j e^\theta i[H_{g,\theta}^\sigma, x_j] - gA_\theta(x_j)$ the term $W_{g,\theta}^{\leqslant\sigma}$ can be written as

$$W_{g,\theta}^{\leqslant\sigma} = ge^\theta i[H_{g,\theta}^\sigma, x] \cdot A_\theta^{\leqslant\sigma}(x) + I, \tag{5.4}$$

where $[H_{g,\theta}^\sigma, x] \cdot A_\theta^{\leqslant\sigma}(x) := \sum_j [H_{g,\theta}^\sigma, x_j] \cdot A_\theta^{\leqslant\sigma}(x_j)$ and $I := \frac{g^2}{2}\left[A_\theta^{\leqslant\sigma}(x) \cdot A_\theta^{\leqslant\sigma}(x) - \Lambda^{\leqslant\sigma} \right]$. Furthermore, using that $[H_{g,\theta}^\sigma, x] \cdot A_\theta^{\leqslant\sigma}(x) = [H_{g,\theta}^\sigma, x \cdot A_\theta^{\leqslant\sigma}(x)] - x \cdot [H_{g,\theta}^\sigma, A_\theta^{\leqslant\sigma}(x)]$, we obtain

$$W_{g,\theta}^{\leqslant\sigma} = g[H_{g,\theta}^\sigma, x \cdot A_\theta^{\leqslant\sigma}(x)] + I + II, \tag{5.5}$$

where $II := -gx \cdot [H_{g,\theta}^\sigma, A_\theta^{\leqslant\sigma}(x)]$. We can now rewrite the operator $\widetilde{R}_{g,\theta}^\sigma W_{g,\theta}^{\leqslant\sigma} \widetilde{R}_{g,\theta}^\sigma$ as

$$gx \cdot A_\theta^{\leqslant\sigma}(x) \widetilde{R}_{g,\theta}^\sigma - g\widetilde{R}_{g,\theta}^\sigma x \cdot A_\theta^{\leqslant\sigma}(x) + \widetilde{R}_{g,\theta}^\sigma (I + II) \widetilde{R}_{g,\theta}^\sigma. \tag{5.6}$$

Let f be a (vector-)function of k. To estimate the expression above we will use the following estimates

$$\left\| a(fG_{x,\theta}^{\leqslant\sigma})^n \psi \right\| \leq \mathrm{C}\sigma^{n/2} \sup_{|k|\leq\sigma} |f| \left\| (H_f^{\leqslant\sigma})^{n/2} \psi \right\|, \quad n = 1, 2, \tag{5.7}$$

$$\left\| a^*(G_{x,\theta}^{\leqslant\sigma}) \psi \right\| \leq \mathrm{C} \left(\sigma^{1/2} \left\| (H_f^{\leqslant\sigma})^{1/2} \psi \right\| + \sigma \|\psi\| \right), \tag{5.8}$$

$$\left\| (H_f^{\leqslant\sigma})^n \widetilde{R}_{g,\theta}^\sigma \right\| \leq \mathrm{C} g^{2(n-1)}, \quad n = 0, 1/2, 1. \tag{5.9}$$

The first two estimates are standard (see e.g. [4, 6]). To prove the last inequality one uses Eqns (4.43) - (4.46). In addition we need the following estimate for any $\delta \ll |\lambda_j - \Sigma|$

$$\left\| (H_f^{\leqslant\sigma})^n e^{-\delta\langle x\rangle} \widetilde{R}_{g,\theta}^{\sigma} e^{\delta\langle x\rangle} \right\| \leq \mathrm{C} g^{2(n-1)}, \quad n = 0, 1/2, 1, \tag{5.10}$$

where, recall, $\Sigma = \inf \sigma_{\mathrm{ess}}(H_p)$, and $\langle x\rangle := \sum_j [1 + x_j^2]^{1/2}$. Eqn (5.10) follows in the same way as (5.9), provided we prove that (4.4) still holds if one replaces $H_{g,\theta}^{\geqslant\sigma}$ and $R_{g,\theta}^{\geqslant\sigma}$ respectively by $e^{-\delta\langle x\rangle} H_{g,\theta}^{\geqslant\sigma} e^{\delta\langle x\rangle}$ and $e^{-\delta\langle x\rangle} R_{g,\theta}^{\geqslant\sigma} e^{\delta\langle x\rangle}$. To prove the latter property, we note that

$$W_{g,\theta}^{\delta,\geqslant\sigma} = W_{g,\theta}^{\geqslant\sigma} + ig\delta e^{-\theta} \sum_j \frac{1}{m_j} \langle x_j\rangle^{-1} x_j \cdot A_\theta^{\geqslant\sigma}(x_j), \tag{5.11}$$

$$H_{0,\theta}^{\delta,\geqslant\sigma} = e^{-\delta\langle x\rangle} H_{p,\theta} e^{\delta\langle x\rangle} + e^{-\theta} H_f^{\geqslant\sigma}, \tag{5.12}$$

where $W_{g,\theta}^{\delta,\geqslant\sigma} := e^{-\delta\langle x\rangle} W_{g,\theta}^{\geqslant\sigma} e^{\delta\langle x\rangle}$ and similarly for $H_{0,\theta}^{\delta,\geqslant\sigma}$, $\overline{P}_\theta^\delta$, $P_{p,j,\theta}^\delta$ and $\overline{P}_{p,j,\theta}^\delta$. Using that $\overline{P}_\theta^\delta = P_{p,j,\theta}^\delta \otimes \overline{P}_\Omega^{\geqslant\sigma} + \overline{P}_{p,j,\theta}^\delta \otimes I$ and the fact that $e^{-\delta\langle x\rangle} H_{p,\theta} e^{\delta\langle x\rangle}$ has the same eigenvalues as $H_{p,\theta}$ we write

$$\begin{aligned} \left[H_{0,\theta}^{\delta,\geqslant\sigma} - z\right]^{-1} \overline{P}_\theta^\delta = &\left[e^{-\theta} H_f^{\geqslant\sigma} + \lambda_j - z\right]^{-1} (P_{p,j,\theta}^\delta \otimes \overline{P}_\Omega^{\geqslant\sigma}) \\ &+ \left[H_{0,\theta}^{\delta,\geqslant\sigma} - z\right]^{-1} (\overline{P}_{p,j,\theta}^\delta \otimes I). \end{aligned} \tag{5.13}$$

Using this decomposition we conclude, similarly to (4.5), that for $\sigma < \mathrm{d}_j \sin|\mathrm{Im}\theta|$ and z in $D(\lambda_j, \sigma/2)$, we have the for some positive constants C_1, C_2

$$\left\| \left[\overline{P}_\theta^\delta H_{0,\theta}^{\delta,\geqslant\sigma} - z\right]^{-1} \left(-W_{g,\theta}^{\delta,\geqslant\sigma} \left[\overline{P}_\theta^\delta H_{0,\theta}^{\delta,\geqslant\sigma} - z\right]^{-1} \right)^n \right\| \leq \mathrm{C}_1 \sigma^{-1} \left(\mathrm{C}_2 g \sigma^{-1/2}\right)^n. \tag{5.14}$$

Now, the first two terms in Eqn (5.6) have only one resolvent each. Using estimates Eqns (5.7) and (5.8), with $n = 1$ and $f \equiv 1$, and Eqns (5.9)-(5.10) with $n = 1/2$, we obtain for these terms, times $e^{-\delta\langle x\rangle}$, with $\delta > 0$, the estimate $O(\sigma^{\frac{1}{2}} + g^{-1}\sigma)$. The operator

$$II = ige^{-\theta} x \cdot (e^{-\theta} p - xgA_\theta^{\leqslant\sigma}(x)) \nabla A_\theta^{\leqslant\sigma}(x) - ge^{-2\theta} x \sum \frac{1}{2m_j} \Delta_j A_\theta^{\leqslant\sigma}(x) + x[H_f^{\leqslant\sigma}, A_\theta^{\leqslant\sigma}(x)] \tag{5.15}$$

has better infrared behaviour than the original operator $A_\theta^{\leqslant\sigma}(x)$ by an extra factor k, ω^2 or ω, which, due to (5.7), with $n = 1$ and $f = \omega$ or k, and (5.9)-(5.10), gives the estimate $\|e^{-\delta\langle x\rangle} \widetilde{R}_{g,\theta}^\sigma II \widetilde{R}_{g,\theta}^\sigma\| = O(g^{-2}\sigma^{\frac{3}{2}})$. Finally, the term I is quadratic in $A_\theta^{\leqslant\sigma}(x)$. Putting it to the normal form and using the estimates (5.7) and (5.9) leads to the estimate $\widetilde{R}_{g,\theta}^\sigma I \widetilde{R}_{g,\theta}^\sigma = O(\sigma)$. Collecting the above estimates and using that $O(\sigma^{\frac{1}{2}} + g^{-1}\sigma + g^{-2}\sigma^{3/2}) = O(g^{-2}\sigma^{3/2})$, we arrive at

$$\left\| e^{-\delta\langle x\rangle} \widetilde{R}_{g,\theta}^\sigma W_{g,\theta}^{\leqslant\sigma} \widetilde{R}_{g,\theta}^\sigma \right\| = O(g^{-2}\sigma^{3/2}). \tag{5.16}$$

Next, we pull $e^{-\delta\langle x\rangle}$, with $\delta > 0$ sufficiently small, from Ψ_j and use the above estimate to obtain

$$\left| (\Psi_{j,\overline{\theta}}, \widetilde{R}_{g,\theta}^\sigma W_{g,\theta}^{\leqslant\sigma} \widetilde{R}_{g,\theta}^\sigma \Psi_{j,\theta}) \right| = O(g^{-2}\sigma^{3/2}).$$

Therefore, the largest contribution to B^1 comes from the term (5.3) that involves the difference of the eigenvalues. Taking into account the factor σ obtained from the z integration yields that $B^1 = O(\sigma^2 g^{-2})$.

One can estimate the operators $B^n,\ n \geq 2$, similarly. In particular, we claim that $B^n = O(\sigma^{\frac{n+4}{2}} g^{-2})$ for $n \geq 2$. Consider for example the term B^2. Since $\widetilde{R}^{\sigma}_{g,\theta} = O(g^{-2})$ and $(\lambda_{j,g} - \lambda^{\geqslant\sigma}_{j,g}) P^{\geqslant\sigma}_{g,\theta} \otimes I = O(g^2\sigma)$, we have that

$$\left\| \widetilde{R}^{\sigma}_{g,\theta}(\lambda_{j,g} - \lambda^{\geqslant\sigma}_{j,g})(P^{\geqslant\sigma}_{g,\theta} \otimes I)\widetilde{R}^{\sigma}_{g,\theta}(\lambda_{j,g} - \lambda^{\geqslant\sigma}_{j,g})(P^{\geqslant\sigma}_{g,\theta} \otimes I)\widetilde{R}^{\sigma}_{g,\theta} \right\| = O(\sigma^2 g^{-2}). \tag{5.17}$$

By pulling $e^{-\delta\langle x\rangle}$ from Ψ_j, we have using (5.16) that

$$\left| (\Psi_{j,\overline{\theta}}, \widetilde{R}^{\sigma}_{g,\theta}(\lambda_{j,g} - \lambda^{\geqslant\sigma}_{j,g}) P^{\geqslant\sigma}_{g,\theta} \otimes I \widetilde{R}^{\sigma}_{g,\theta} W^{\leqslant\sigma}_{g,\theta} \widetilde{R}^{\sigma}_{g,\theta} \Psi_{j,\theta}) \right| = O(\sigma^{5/2} g^{-2}). \tag{5.18}$$

Using (5.6), we have

$$\begin{aligned} \widetilde{R}^{\sigma}_{g,\theta} W^{\leqslant\sigma}_{g,\theta} \widetilde{R}^{\sigma}_{g,\theta} W^{\leqslant\sigma}_{g,\theta} \widetilde{R}^{\sigma}_{g,\theta} = & \underbrace{g \widetilde{R}^{\sigma}_{g,\theta} W^{\leqslant\sigma}_{g,\theta} x \cdot A^{\leqslant\sigma}_{\theta}(x) \widetilde{R}^{\sigma}_{g,\theta}}_{III} - \underbrace{g \widetilde{R}^{\sigma}_{g,\theta} W^{\leqslant\sigma}_{g,\theta} \widetilde{R}^{\sigma}_{g,\theta} x \cdot A^{\leqslant\sigma}_{\theta}(x)}_{IV} \\ & + \underbrace{\widetilde{R}^{\sigma}_{g,\theta} W^{\leqslant\sigma}_{g,\theta} \widetilde{R}^{\sigma}_{g,\theta} I \widetilde{R}^{\sigma}_{g,\theta}}_{V} + \underbrace{\widetilde{R}^{\sigma}_{g,\theta} W^{\leqslant\sigma}_{g,\theta} \widetilde{R}^{\sigma}_{g,\theta} II \widetilde{R}^{\sigma}_{g,\theta}}_{VI}. \end{aligned} \tag{5.19}$$

It follows from (5.7) and (5.9)-(5.10) and from (5.8) that

$$\left\| e^{-\delta\langle x\rangle} III \right\| = O(\sigma) \text{ and } \left\| e^{-\delta\langle x\rangle} IV \right\| = O(\sigma + \sigma^{3/2} g^{-1}). \tag{5.20}$$

Since the operator I is quadratic in $A^{\leqslant\sigma}_{\theta}$, we obtain by putting it to the normal form and using again (5.7) and (5.9)-(5.10)

$$\left\| e^{-\delta\langle x\rangle} V \right\| = O(\sigma^{3/2}). \tag{5.21}$$

Finally, as above the fact than II has better infrared behavior than $A^{\leqslant\sigma}_{\theta}$ by the factor ω leads to

$$\left\| e^{-\delta\langle x\rangle} VI \right\| = O(\sigma^2 g^{-2}). \tag{5.22}$$

By pulling $e^{-\delta\langle x\rangle}$ from Ψ_j and using (5.19)-(5.22) we find that

$$(\Psi_{j,\overline{\theta}}, \widetilde{R}^{\sigma}_{g,\theta} W^{\leqslant\sigma}_{g,\theta} \widetilde{R}^{\sigma}_{g,\theta} W^{\leqslant\sigma}_{g,\theta} \widetilde{R}^{\sigma}_{g,\theta} \Psi_{j,\theta}) = O(\sigma^2 g^{-2}). \tag{5.23}$$

Together with a factor of σ obtained from the z integration, (5.17), (5.18) and (5.23) yield the estimate $B^2 = O(\sigma^3 g^{-2})$.

Instead of (3.40), Section 3, we have in the case of non-relativistic QED

$$\left(\Psi_j, e^{-itH_g} f(H_g) \Psi_j\right) = e^{-it\lambda_{j,g}} + O(g^2\sigma^{-1}) + O(\sigma^2 g^{-2}), \quad t \geq 0.$$

Optimizing and removing the f dependence as in the proof of Theorem 1.1 gives

$$\left(\Psi_j, e^{-itH_g} \Psi_j\right) = e^{-it\lambda_{j,g}} + O(g^{2/3}), \quad t \geq 0.$$

6 Proof of Theorem 1.4

Let P_Ω be the projection on the vacuum Ω in $\mathcal{F}_s$. We prove Theorem 1.4 for the set $\mathcal{D}'$ chosen explicitly as

$$\mathcal{D}' := \{\psi \in \mathcal{D} | \ \ \|d\Gamma(\omega^{-1/2})(1-P_\Omega)\psi\| < \infty\}$$

for the Nelson model and as

$$\mathcal{D}' := \{\psi \in \mathcal{D} | \ \ \|e^{\delta\langle x\rangle} d\Gamma(\omega^{-1/2})(1-P_\Omega)\psi\| < \infty \text{ for some } \delta > 0\}$$

for the QED one. Since $\mathcal{U}_\theta d\Gamma(\omega^{-1/2}) = e^{\theta/2} d\Gamma(\omega^{-1/2})\mathcal{U}_\theta$, the set $\mathcal{D}'$ is dense in $\mathcal{D}$.

We conduct the proof for the Nelson model only. To extend it to the QED one uses the methodology of Section 5. As in Sections 2 - 4, the symbol $H_{g,\theta}$ stands for the dilatation transformation, (1.10) of the Nelson Hamiltonian $H_g = H_g^N$.

The RG analysis [4, 5] shows that given $\delta > 0$, there exist $g_* > 0$ and $\varphi_* \in (0, \varphi_0)$ s.t. for $g \le g_*$ and $\mathrm{Im}\theta \in (\varphi_*, \varphi_0)$, the spectrum of the operator $H_{g,\theta}$ in the half-plane $\{\mathrm{Re}z \le \Sigma - \delta\}$ lies in the union of wedges

$$S_j := \lambda_{j,g} + \{z \in \mathbb{C} | \ \ |\arg(z) - \mathrm{Im}\theta| \le \epsilon\},$$

where $\lambda_{j,g} = \lambda_j + O(g^2)$, $\mathrm{Im}\lambda_{j,g} \le 0$ and $\epsilon < |\mathrm{Im}\theta|$ is a positive number [5]. Moreover, the apices, $\lambda_{j,g}$, of these wedges are the eigenvalues of $H_{g,\theta}$. If, in addition, condition (C) holds for λ_j then $\mathrm{Im}\lambda_{j,g} \le -\text{const. } g^2$.

We take $z \in W_{\lambda_{j,g}}^{\varphi_1,\varphi_2}$ with $\varphi_1 = \pi/2 - \varphi_0$ and $\varphi_2 > 3\pi/2 - \varphi_*$. We want to estimate $(\psi, (H_{g,\theta} - z)^{-1}\psi)$. Using an infrared cut-off as in section 2, we decompose

$$H_{g,\theta} = \widetilde{H}_{g,\theta}^\sigma + \widetilde{W}_{g,\theta}^{\leqslant\sigma}, \tag{6.1}$$

see (2.14). The infrared cut-off Hamiltonian $\widetilde{H}_{g,\theta}^\sigma$ has an eigenvalue at $\lambda_{j,g}$. We use the second resolvent equation

$$(H_{g,\theta} - z)^{-1} = (\widetilde{H}_{g,\theta}^\sigma - z)^{-1} + (\widetilde{H}_{g,\theta}^\sigma - z)^{-1}\widetilde{W}_{g,\theta}^{\leqslant\sigma}(H_{g,\theta} - z)^{-1}. \tag{6.2}$$

Let $\widetilde{R}_{g,\theta}^\sigma(z) := (\widetilde{H}_{g,\theta}^\sigma - z)^{-1}$ and let $P_\Omega^{\leqslant\sigma}$ be the projection onto the vacuum state of $\mathcal{F}_s^{\leqslant\sigma}$ and $\overline{P} = 1 - P$. Then

$$\widetilde{R}_{g,\theta}^\sigma(z) = \frac{P_{g,\theta}^{\geqslant\sigma} \otimes P_\Omega^{\leqslant\sigma}}{\lambda_{j,g} - z} + \frac{P_{g,\theta}^{\geqslant\sigma} \otimes \overline{P}_\Omega^{\leqslant\sigma}}{\lambda_{j,g} + e^{-\theta}H_f^{\leqslant\sigma} - z} + \widehat{R}_{g,\theta}^\sigma(z), \tag{6.3}$$

where, as above,

$$\widehat{R}_{g,\theta}^\sigma(z) := (\overline{P}_{g,\theta}^{\geqslant\sigma} \otimes I)\widetilde{R}_{g,\theta}^\sigma(z). \tag{6.4}$$

By our condition on z we can pick θ so that

$$\mathrm{Re}(e^\theta(\lambda_{j,g} - z)) \ge 0, \tag{6.5}$$

i.e. $|\mathrm{Im}\theta + \arg(\lambda_{j,g} - z)| \le \pi/2$. Then

$$|(\psi, \frac{P_{g,\theta}^{\geqslant\sigma} \otimes \overline{P}_\Omega^{\leqslant\sigma}}{\lambda_{j,g} + e^{-\theta}H_f^{\leqslant\sigma} - z}\psi)| \le \|(H_f^{\leqslant\sigma})^{-1/2}\overline{P}_\Omega^{\leqslant\sigma}\psi\|^2. \tag{6.6}$$

[5]The proof for the QED model without the confinement assumption is given in [7].

(More generally, the l.h.s. is bounded by $|\lambda_{j,g} - z|^{-\alpha}\|(H_f^{\leqslant\sigma})^{-(1-\alpha)/2}\overline{P}_\Omega^{\leqslant\sigma}\psi\|^2$ for $0 \le \alpha \le 1$.) Furthermore, an elementary analysis of the $n-$photon sectors shows that

$$\|(H_f^{\leqslant\sigma})^{-1/2}\overline{P}_\Omega^{\leqslant\sigma}\psi\| \le \|d\Gamma(\omega^{-1/2})\overline{P}_\Omega\psi\|. \tag{6.7}$$

Hence, by the definition of $\mathcal{D}'$, we have that, for all $\psi \in \mathcal{D}'$,

$$|(\psi, \frac{P_{g,\theta}^{\geqslant\sigma} \otimes \overline{P}_\Omega^{\leqslant\sigma}}{\lambda_{j,g} + e^{-\theta}H_f^{\leqslant\sigma} - z}\psi)| \le \mathrm{C}. \tag{6.8}$$

Next, to estimate $\widehat{R}_{g,\theta}^\sigma(z)$, see Eq. (6.4), we use the representation (4.44). Applying to $H_{g,\theta}^{\geqslant\sigma}$ a renormalization group analysis as in [4, 5, 12, 14], one can show that the spectrum of $\widetilde{H}_{g,\theta}^{\geqslant\sigma}$ is of the form pictured in Figure 3, and that for $|z - \lambda_{j,g}| \le \sigma/2$ and $\omega \ge 0$

$$\|\widehat{R}_{g,\theta}^{\geqslant\sigma}(z - e^{-\theta}\omega)\| \le \mathrm{C}(\sigma + \omega)^{-1}, \tag{6.9}$$

which, together with (4.44), implies, for $|z - \lambda_{j,g}| \le \sigma/2$ and $n = 0, 1/2, 1$, the estimate

$$\|(H_f^{\leqslant\sigma})^n\widehat{R}_{g,\theta}^\sigma(z)\| \le \mathrm{C}\sigma^{n-1}, \tag{6.10}$$

for some constant C.

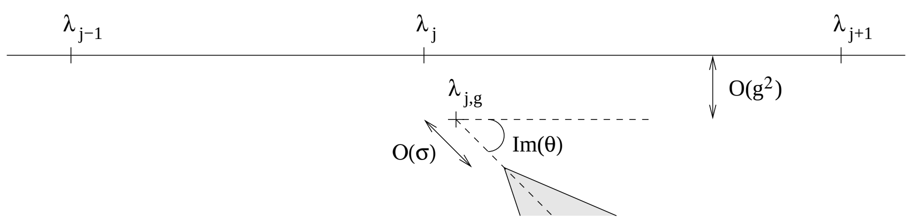

Figure 3: **Spectrum of $\widetilde{H}_{g,\theta}^{\geqslant\sigma}$ near $\lambda_{j,g}$**

Eqns (6.3), (6.8) and (6.10) imply that, for $\psi \in \mathcal{D}'$,

$$|(\psi, (\widetilde{R}_{g,\theta}^\sigma(z) - \frac{P_{g,\theta}^{\geqslant\sigma} \otimes P_\Omega^{\leqslant\sigma}}{\lambda_{j,g} - z})\psi)| \le \mathrm{C}/\sigma, \tag{6.11}$$

Finally we estimate the last term on the r.h.s. Eq. (6.2). Recall that

$$\widetilde{W}_{g,\theta}^{\leqslant\sigma} = W_{g,\theta}^{\leqslant\sigma} - \left(\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}\right)\mathcal{V}^{-1}\left(P_{g,\theta}^{\geqslant\sigma} \otimes I\right)\mathcal{V}, \tag{6.12}$$

where

$$W_{g,\theta}^{\leqslant\sigma} := W_{g,\theta} - W_{g,\theta}^{\geqslant\sigma} = g\phi(G_{x,\theta}^{\leqslant\sigma}). \tag{6.13}$$

Below, we let $\sigma \to 0$, as $|\lambda_{j,g} - z| \to 0$. Hence we have to estimate $\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}$ for any $\sigma > 0$. We claim that

$$|\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}| = O\left((g\sigma^{1/2+\mu})^2\right). \tag{6.14}$$

This estimate is proven in the proposition at the end of this section.

Iterating the last term on the r.h.s. of Eq. (6.2) we see that the worst term is $\widetilde{R}^{\sigma}_{g,\theta}(z)\widetilde{W}^{\leqslant\sigma}_{g,\theta}\widetilde{R}^{\sigma}_{g,\theta}(z)$. We use the decomposition (6.3). Since the operator $W^{\leqslant\sigma}_{g,\theta}$ is in normal form, we see that the term coming from sandwiching it between the first term on the r.h.s. of (6.3) vanishes. Thus, it remains to consider the terms

$$\widetilde{R}^{\sigma}_{g,\theta}(z)\left(\lambda_{j,g}-\lambda^{\geqslant\sigma}_{j,g}\right)\left(P^{\geqslant\sigma}_{g,\theta}\otimes I\right)\widetilde{R}^{\sigma}_{g,\theta}(z), \tag{6.15}$$

$$[\frac{P^{\geqslant\sigma}_{g,\theta}\otimes P^{\leqslant\sigma}_{\Omega}}{\lambda_{j,g}-z}+\frac{P^{\geqslant\sigma}_{g,\theta}\otimes\overline{P}^{\leqslant\sigma}_{\Omega}}{\lambda_{j,g}+e^{-\theta}H^{\leqslant\sigma}_{f}-z}+\widehat{R}^{\sigma}_{g,\theta}(z)]$$

$$\times W^{\leqslant\sigma}_{g,\theta}[\frac{P^{\geqslant\sigma}_{g,\theta}\otimes\overline{P}^{\leqslant\sigma}_{\Omega}}{\lambda_{j,g}+e^{-\theta}H^{\leqslant\sigma}_{f}-z}+\widehat{R}^{\sigma}_{g,\theta}(z)] \tag{6.16}$$

and the term obtained by switching the right and left factors in (6.17).

We note that, by the decomposition (6.3) and the definition of $\widehat{R}^{\sigma}_{g,\theta}(z)$, Eq. (6.15) can be written as

$$\left(\lambda_{j,g}-\lambda^{\geqslant\sigma}_{j,g}\right)\frac{P^{\geqslant\sigma}_{g,\theta}\otimes P^{\leqslant\sigma}_{\Omega}}{(\lambda_{j,g}-z)^2}+\left(\lambda_{j,g}-\lambda^{\geqslant\sigma}_{j,g}\right)\frac{P^{\geqslant\sigma}_{g,\theta}\otimes\overline{P}^{\leqslant\sigma}_{\Omega}}{(\lambda_{j,g}+e^{-\theta}H^{\leqslant\sigma}_{f}-z)^2}. \tag{6.17}$$

Using (6.14) we obtain the following estimate for (6.15):

$$(6.15)=O\left((g\sigma^{1/2+\mu}|\lambda_{j,g}-z|^{-1})^2\right). \tag{6.18}$$

To estimate (6.16), we first observe that, due to (6.5), we have that, for $n=0,1/2,1$

$$\|(H^{\leqslant\sigma}_{f})^n(\lambda_{j,g}+e^{-\theta}H^{\leqslant\sigma}_{f}-z)^{-1}\|\leq \mathrm{C}|\lambda_{j,g}-z|^{n-1}. \tag{6.19}$$

Assume $\sigma\geq|z-\lambda_{j,g}|$. Using estimates (2.7), (6.10) and (6.19) (or (6.25)), the fact that $P^{\leqslant\sigma}_{\Omega}a^*(G^{\leqslant\sigma}_{x,\theta})=0$ and standard estimates on the creation and annihilation operators, and remembering the condition that $\mathrm{Re}(e^{\theta}(\lambda_{j,g}-z))\geq 0$, we obtain the bound

$$\|\widetilde{R}^{\sigma}_{g,\theta}(z)W^{\leqslant\sigma}_{g,\theta}\widetilde{R}^{\sigma}_{g,\theta}(z)\|\leq \mathrm{C}\frac{1}{|z-\lambda_{j,g}|}g\sigma^{\frac{1}{2}+\mu}(\frac{1}{|z-\lambda_{j,g}|^{1/2}}+\frac{1}{\sigma^{1/2}}). \tag{6.20}$$

This together with (6.18) yields

$$\|\widetilde{R}^{\sigma}_{g,\theta}(z)\widetilde{W}^{\leqslant\sigma}_{g,\theta}\widetilde{R}^{\sigma}_{g,\theta}(z)\|\leq \mathrm{C}\frac{g\sigma^{\frac{1}{2}+\mu}}{|z-\lambda_{j,g}|}(\frac{1}{\sigma^{1/2}}+\frac{1}{|z-\lambda_{j,g}|^{1/2}}+\frac{g\sigma^{\frac{1}{2}+\mu}}{|z-\lambda_{j,g}|}). \tag{6.21}$$

Since, as we mentioned, the higher order iterates of (6.2) are estimated similarly and lead to improved estimates, we conclude, assuming $\sigma\geq|z-\lambda_{j,g}|$, that

$$\|\widetilde{R}^{\sigma}_{g,\theta}(z)\widetilde{W}^{\leqslant\sigma}_{g,\theta}R_{g,\theta}(z)\|\leq \mathrm{C}(\frac{g\sigma^{\frac{1}{2}+\mu}}{|z-\lambda_{j,g}|^{3/2}}+\frac{g^2\sigma^{1+2\mu}}{|z-\lambda_{j,g}|^2}), \tag{6.22}$$

where $R_{g,\theta}(z):=(H_{g,\theta}-z)^{-1}$.

It follows from (6.2), (6.11) and (6.22) that, for g small enough,

$$|(\psi,((H_{g,\theta}-z)^{-1}-\frac{1}{\lambda_{j,g}-z}P_{g,\theta}^{\geqslant\sigma}\otimes P_\Omega^{\leqslant\sigma})\psi)|\leq \mathrm{C}(\frac{1}{\sigma}+\frac{g\sigma^\alpha}{r^{3/2}}+\frac{g^2\sigma^{2\alpha}}{r^2}), \tag{6.23}$$

where $r:=|z-\lambda_{j,g}|$ and $\alpha:=1/2+\mu$, for some constant C, provided $\sigma\geq|z-\lambda_{j,g}|$. We now pick $\sigma=r^\beta g^{-(3/2+\mu)^{-1}}$, where $\beta:=(1+\frac{2}{3}\mu)^{-1}$. By our assumption, $\beta<1$ and therefore $\sigma>r$. Then, for this choice of σ,

$$|(\psi,((H_{g,\theta}-z)^{-1}-\frac{1}{\lambda_{j,g}-z}P_{g,\theta}^{\geqslant\sigma}\otimes P_\Omega^{\leqslant\sigma})\psi)|\leq \mathrm{C}g^{(3/2+\mu)^{-1}}r^{-\beta}.$$

Let $\psi_\theta:=\mathcal{U}_\theta\psi$. The last estimate, together with the relation

$$(\psi,(H_g-z)^{-1}\psi)=(\psi_{\bar\theta},(H_{g,\theta}-z)^{-1}\psi_\theta), \tag{6.24}$$

implies (i) and (ii) in Theorem 1.4, with $\beta=(1+\frac{2}{3}\mu)^{-1}$. $\square$

Remark 6.1 *The expression for β can be improved if one uses* (6.7) *to conclude that, for $n=0,1/2$,*

$$\|(H_f^{\leqslant\sigma})^n(\lambda_{j,g}+e^{-\theta}H_f^{\leqslant\sigma}-z)^{-1}\psi\|\leq \mathrm{C}|\lambda_{j,g}-z|^{n-1/2}\|d\Gamma(\omega^{-1/2})\overline{P}_\Omega\psi\|, \tag{6.25}$$

which is better than (6.19)*. This estimate leads to the inequality*

$$|(\psi,\widetilde{R}_{g,\theta}^\sigma(z)W_{g,\theta}^{\leqslant\sigma}\widetilde{R}_{g,\theta}^\sigma(z)\psi)|\leq \mathrm{C}\frac{1}{|z-\lambda_{j,g}|}g\sigma^{\frac{1}{2}+\mu}(\frac{\sigma^{1/2}}{|z-\lambda_{j,g}|^{1/2}}+\frac{1}{\sigma^{1/2}}), \tag{6.26}$$

which has a better r.h.s than (6.20)*.*

Remark 6.2 *To define resonances for the QED model it is technically more convenient to use a family of unitary transformations different from the dilatation one (see [7]).*

Proposition 6.3 *Under the conditions of Theorem 1.4, we have for any $\sigma>0$*

$$\lambda_{j,g}-\lambda_{j,g}^{\geqslant\sigma}=O(g^2\sigma^{1+\mu}). \tag{6.27}$$

Proof. To prove (6.14) we use the RG approach. Here we only point out particularities of the present problem and outline the general strategy; technical details can be found in [4, 5, 12, 14] (see also [7] for the QED case). Since we do not go into details, we use the Feshbach-Schur map, rather than the smooth Feshbach-Schur map, to underpin our construction. The former ([4, 5]) is simpler to formulate but the latter ([12, 13, 14]) is easier to handle technically. Our strategy follows ([14]).

First we apply the Feshbach-Schur map $\mathcal{F}_{P_{\rho_0}}$ associated to the projection $P_\rho:=P_{g,\theta}^{\geqslant\sigma}\otimes\chi_\rho^{\leqslant\sigma}$, where $\chi_\rho^{\leqslant\sigma}:=\chi_{H_f^{\leqslant\sigma}\leqslant\rho}$. For $z\in D(\lambda_{j,g}^{\geqslant\sigma},\sigma/2)$ and $\rho_0=\sigma$, the operator $H_{g,\theta}-z$ is in the domain of $\mathcal{F}_{P_{\rho_0}}$. Indeed, an easy estimate shows that the operator $\overline{P}_{\rho_0}H_{g,\theta}^\sigma\overline{P}_{\rho_0}-z$ is invertible on $\mathrm{Ran}\overline{P}_{\rho_0}$ and $\|[H_f^{\leqslant\sigma}+\sigma]\overline{P}_{\rho_0}\left[\overline{P}_{\rho_0}H_{g,\theta}^\sigma\overline{P}_{\rho_0}-z\right]^{-1}\overline{P}_{\rho_0}\|\leq \mathrm{C}$. Since $\|[H_f^{\leqslant\sigma}+\sigma]^{-1/2}W_{g,\theta}^{\leqslant\sigma}[H_f^{\leqslant\sigma}+\sigma]^{-1/2}\|\leq \mathrm{C}g\sigma^\mu$, we see by Neumann series expansion that the operator $\overline{P}_{\rho_0}H_{g,\theta}\overline{P}_{\rho_0}-z$ is invertible on $\mathrm{Ran}\overline{P}_{\rho_0}$ and $\|\overline{P}_{\rho_0}\left[\overline{P}_{\rho_0}H_{g,\theta}\overline{P}_{\rho_0}-z\right]^{-1}\overline{P}_{\rho_0}\|\leq \mathrm{C}/\sigma$. Hence the operator $H_{g,\theta}-z$ is in the domain of $\mathcal{F}_{P_{\rho_0}}$, as claimed. Next, we note that

$$\mathcal{F}_{P_{\rho_0}}(H_{g,\theta}-z)=P_{g,\theta}^{\geqslant\sigma}\otimes H_z,$$

where the operator H_z acts on $\mathrm{Ran}\chi_{\rho_0}^{\leqslant\sigma} \subset \mathcal{F}_s^{\leqslant\sigma}$ and is given by

$$H_z := \chi_{\rho_0}^{\leqslant\sigma}(\psi_{g,\theta}^{\geqslant\sigma}, \big(\lambda_{j,g}^{\geqslant\sigma} - z + H_f^{\leqslant\sigma} + W_{g,\theta}^{\leqslant\sigma} + U\big)\psi_{g,\theta}^{\geqslant\sigma})\chi_{\rho_0}^{\leqslant\sigma}, \tag{6.28}$$

where $U := -W_{g,\theta}^{\leqslant\sigma}\overline{P}_{\rho_0}\left[\overline{P}_{\rho_0}H_{g,\theta}\overline{P}_{\rho_0} - z\right]^{-1}\overline{P}_{\rho_0}W_{g,\theta}^{\leqslant\sigma}$.

By the isospectrality of the Feshbach-Schur map (see [4, 5, 6, 12, 14]), we have that $z \in D(\lambda_{j,g}^{\geqslant\sigma}, \sigma/2)$ is an eigenvalue of $H_{g,\theta}$ iff 0 is an eigenvalue of H_z. To investigate the spectral properties of H_z, we make use of the renormalization group method.

As a first step, we rewrite the operator H_z in a generalized normal form. To this end we expand the resolvent on the r.h.s. in a Neumann series in $W_{g,\theta}^{\leqslant\sigma}$ and normal order the creation and annihilation operators not entering the expression for $H_f^{\leqslant\sigma}$. This brings the operator H_z to the form (see [4, 5, 12, 14])

$$H_z := \chi_{\rho_0}^{\leqslant\sigma}(E_z + T_z + W_z)\chi_{\rho_0}^{\leqslant\sigma}, \tag{6.29}$$

where E_z is a number (more precisely, a complex function of z and other parameters), T_z is a differentiable function of $H_f^{\leqslant\sigma}$ and W_z is an operator in the generalized normal form that is a sum of terms with at least one creation or annihilation operator. A standard computation gives that $E_z := \lambda_{j,g}^{\geqslant\sigma} - z + \Delta E_z$, with

$$\Delta E_z := -\int (\psi_{g,\theta}^{\geqslant\sigma}, G_{x,\theta}^{\leqslant\sigma}(k)\overline{P}_{g,\theta}^{\geqslant\sigma}\left[\overline{P}_{g,\theta}^{\geqslant\sigma}H_{g,\theta}^{\geqslant\sigma}\overline{P}_{g,\theta}^{\geqslant\sigma} + e^{-\theta}\omega - z\right]^{-1}\overline{P}_{g,\theta}^{\geqslant\sigma}G_{x,\theta}^{\leqslant\sigma}(k)\psi_{g,\theta}^{\geqslant\sigma})dk + h.o.t.,$$

$$T_z := H_f^{\leqslant\sigma} - \int (\psi_{g,\theta}^{\geqslant\sigma}, G_{x,\theta}^{\leqslant\sigma}(k)f(H_f^{\leqslant\sigma} + \omega)G_{x,\theta}^{\leqslant\sigma}(k)\psi_{g,\theta}^{\geqslant\sigma})dk + h.o.t.,$$

$$W_z := (\psi_{g,\theta}^{\geqslant\sigma}, \big(W_{g,\theta}^{\leqslant\sigma} - \int\int G_{x,\theta}^{\leqslant\sigma}(k)a^*(k)f(H_f^{\leqslant\sigma} + \omega + \omega')a(k')G_{x,\theta}^{\leqslant\sigma}(k')dkdk'\big)\psi_{g,\theta}^{\geqslant\sigma}) + h.o.t.,$$

where $f(H_f^{\leqslant\sigma}) := \overline{P}_{\rho_0}\left[\overline{P}_{\rho_0}(H_{g,\theta}^{\geqslant\sigma} + e^{-\theta}H_f^{\leqslant\sigma})\overline{P}_{\rho_0} - z\right]^{-1}\overline{P}_{\rho_0}$. Clearly,

$$\Delta E_z = O\left((g\sigma^{1/2+\mu})^2\right) \text{ and } \chi_{\rho_0}^{\leqslant\sigma}W_z\chi_{\rho_0}^{\leqslant\sigma} = O\left(g\sigma^{1+\mu}\right). \tag{6.30}$$

Let $a^{\#}(k)$ stand for either $a(k)$ or $a^*(k)$, $k \in \mathbb{R}^3$. We define the *scaling transformation* S_ρ : $\mathcal{B}[\mathcal{F}_s^{\leqslant\sigma}] \to \mathcal{B}[\mathcal{F}_s^{\leqslant\sigma/\rho}]$, by

$$S_\rho(I) := I, \quad S_\rho(a^{\#}(k)) := \rho^{-3/2}\, a^{\#}(\rho^{-1}k), \tag{6.31}$$

and the dilatation transform, by $A_\rho(A) := \rho^{-1}A$. Now we rescale the operator H_z as $H_z^{(0)} := A_\sigma(S_\sigma(H_z))$. The new operator acts on $\mathrm{Ran}\chi_1^{\leqslant 1} \subset \mathcal{F}_s^{\leqslant 1}$. The last estimate in (6.30) and an estimate on the derivative of T_z as a function of $H_f^{\leqslant\sigma}$, which we do not display here, show that the operator $H_z^{(0)}$ is in the domain of the Feshbach-Schur map $\mathcal{F}_{\chi_\rho^{\leqslant\sigma}}$, provided $1/2 \geq \rho \gg g\sigma^\mu$ and $\rho \gg |E_z|/\sigma$ (the latter inequality is also considered as a restriction on z).

If we neglect the term W_z in $H_z^{(0)}$ (see (6.29)) then the remaining operator has the vacuum Ω as an eigenvector corresponding to the eigenvalue 0, provided z solves the equation $E_z^{(0)} := (\Omega, H_z^{(0)}\Omega) = E_z/\sigma = 0$. One can show ([14]) that this equation has a unique solution $\lambda_{j,g}^{(1)} =$

$\lambda_{j,g}^{\geqslant\sigma} + O\left((g\sigma^{1/2+\mu})^2\right)$. By the isospectrality mentioned above, this is our first approximation to $\lambda_{j,g}$.

Now we introduce the decimation map $F_\rho := \mathcal{F}_{\chi_\rho^{\leqslant\sigma}}$. On the domain of the decimation map F_ρ we define the renormalization map $\mathcal{R}_\rho$ as [6]

$$\mathcal{R}_\rho := A_\rho \circ S_\rho \circ F_\rho. \tag{6.32}$$

By the above, the operator $H_z^{(0)}$ is in the domain of the decimation map F_ρ and therefore in the domain the renormalization map $\mathcal{R}_\rho$, provided $1/2 \geq \rho \gg g\sigma^\mu$ and $\rho \gg |E_z|/\sigma$. Iterating this map as in [14] we obtain a sequence of operators $H_z^{(n)},\ n = 0, 1, 2, ...,$ (Hamiltonians on scales $0, 1, ...$) acting on the space $\mathrm{Ran}\chi_1^{\leqslant 1} \subset \mathcal{F}_s^{\leqslant 1}$. Again, one argues that 0 is an approximate eigenvalue of the operators $H_z^{(n)}$, provided z satisfies the equations $E_z^{(n)} := (\Omega, H_z^{(n)}\Omega) = 0$. Namely, one proves that the equations $(\Omega, H_z^{(n)}\Omega) = 0$ in z have have unique solutions $\lambda_{j,g}^{(n)}$ satisfying

$$\lambda_{j,g}^{(n)} = \lambda_{j,g}^{\geqslant\sigma} + O\left((g\sigma^{1/2+\mu})^2\right)$$

and $|\lambda_{j,g}^{(n)} - \lambda_{j,g}^{(n-1)}| \leq const\ \rho^n$ (see [14], Proposition V.3). Consequently, $\lambda_{j,g}^{(n)}$ converge, $\lambda_{j,g}^{(n)} \to \lambda_{j,g}$, as $n \to \infty$. By the isospectrality of $\mathcal{R}_\rho$ we conclude that the operator $H_z^{(0)}$ has a simple eigenvalue 0, provided $z = \lambda_{j,g}$ (see [14], Theorem V.2). Hence, by the isospectrality of the Feshbach-Schur map, the operator $H_{g,\theta}$ has a unique eigenvalue $\lambda_{j,g}$ in the disc $D(\lambda_{j,g}^{\geqslant\sigma}, \sigma/2)$ and this eigenvalue satisfies (6.14). Since on the other hand $\lambda_{j,g}^{\geqslant\sigma} = \lambda_j + O(g^2)$ is the unique eigenvalue of the operator $H_{g,\theta}^{\geqslant\sigma}$ bifurcating from the eigenvalue λ_j of H_0, we conclude that $\lambda_{j,g}$ is the unique eigenvalue of the operator $H_{g,\theta}$ emerging from the eigenvalue λ_j of H_0. $\Box$

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The research of these authors is supported by NSERC under Grant # 7901.
∗ Supported by the Centre for Theory in Natural Science